Lecture 25
April 23, 2013

Reference: Osborne + Rubenstein, Chap. 13 (and Myersen Sect. 9.3)

Coalitions in Cooperative Games

Def A coalitional game with transferable payoff (or utilities) consists of $\langle N, v \rangle$

N = finite set of players.

called the characteristic function of the game.

$(v(S) : S \subseteq N)$ $\quad v(S) \in \mathbb{R}$ $\quad v(S)$ is the worth of coalition S.

(set $v(\emptyset) = 0$)

Interpretation In a strategic form games often a group of players can get better payoffs for themselves by cooperation. $v(S)$ represents what the players within S can achieve on their own. The set N itself is called the grand coalition. A question addressed in this theory is when there exists transfer payments so players have incentive to cooperate in grand coalition.

<u>Definition</u> $\langle N, v \rangle$ is <u>cohesive</u> if

$$v(N) \geq \sum_{k=1}^{K} v(S_k) \quad \text{for every partition } (S_1, \dots, S_K) \text{ of } N.$$

Henceforth we restrict attention to cohesive characteristic functions.

<u>Def</u> A vector of payoffs, $(x_i : i \in S)$ for a coalition S is <u>S-feasible</u> if $\overbrace{\sum_{i \in S} x_i}^{x(S')} = v(S)$

A vector of N-feasible payoffs is called <u>feasible</u>.

<u>Def</u> The <u>core</u> of the coalitional game $\langle N, v \rangle$ with transferable payments is the set of $(x_i)_{i \in N} \in \mathbb{R}^N$ such that

(1) (x_i) is feasible

(2) $\sum_{i \in S} x_i \geq v(S)$ for all i (i.e. can't be beat in S by an S-feasible payoff vector.)

<u>Example</u> $N = \{1,2,3\}$ (3-player majority game.)

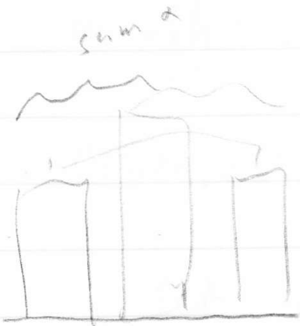
$v(\{i\}) = 0$ all i

$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = \alpha$

$v(\{1,2,3\}) = 1$ (where $0 \le \alpha \le 1$ is fixed)

Core x_1, x_2, x_3 Need $x_i \ge v(\{i\}) = 0$

$x_i + x_j \ge v(\{i,j\}) = \alpha$ $i \ne j$



$x_1 + x_2 + x_3 = \alpha > 0$

$2\alpha - x_1 - x_1 \ge \alpha$

$2x_1 \le \alpha$

$x_1 \le \alpha/2$

If x is in the core, then

$1 = x_1 + x_2 + x_3 \ge \frac{1}{2}\{(x_1+x_2) + (x_1+x_3) + (x_2+x_3)\}$

$\ge \frac{3}{2}\alpha$ $\therefore$ Core is empty if $\alpha > 2/3$

If $0 \le \alpha \le 2/3$ then the core is not empty. e.g. vectors

$(1-\alpha, \frac{\alpha}{2}, \frac{\alpha}{2})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are in core

if $\alpha \le 2/3$

Example Production economy

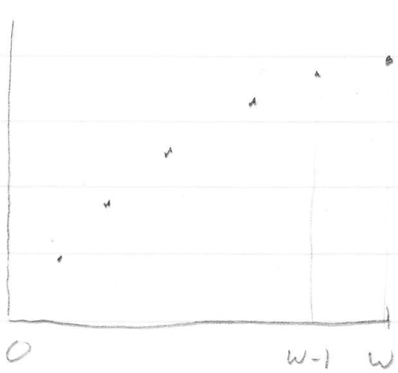
c = capitalist, owns factory
W = set of workers

If capitalist cooperates and m workers work, worth is $f(m)$, where $f(0)=0$, $f:\mathbb{R}_+\to\mathbb{R}_+$ is concave.

set $N = \{c\} \cup W$

$$v(S)=\begin{cases}0 & \text{if } c\notin S\\ f(|S\cap W|)\end{cases}$$

Let's find core. (Answer: $\{x\in\mathbb{R}^n : 0\le x_i \le f(w)-f(w-1)$,
$x_c + x_1 + \cdots + x_w = f(w)$.



0 $\quad$ w-1 w

$f(w)-f(w-1)$ – marginal value of last worker added.

Core assignment $(x_c, x_1, \ldots, x_w)$

(*) Need $x_i \geq v(\{i\}) = 0$ for all i.

(**) Need $x_j \leq f(w) - f(w-1)$ for all $i\neq c$, else
$$v(N-j) > v(N) - x_j = \sum_{i\in N-j} x_i .$$

Conditions (*) + (**) + feasibility are sufficient i.e.

<u>C5</u> Function v is called convex (supermodular)

if $\quad v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad$ (still $v(\emptyset) = \emptyset$)

Let $S_i = \{1, \cdots i\}$ $1 \leq i \leq N$ (could use <u>any</u>

increasing sequence of coalitions) Then,

if v is convex: $x_i = v(S_i) - v(S_{i-1})$

defines a vector in the core.

<u>Proof</u> Need for any coalition R,

$$x(R) \triangleq \sum_i x_i \geq v(R)$$

<u>Let</u> $R = \{i_1, \cdots i_g\}$ with $i_1 < i_2 \cdots < i_g$.

Then $$v(R) = \sum_{j=1}^{g} v(\{i_1, \cdots, i_j\}) - v(\{i_1, \cdots, i_{j-1}\})$$

$$\geq \sum_{j=1}^{g} \underbrace{v(\{i_j\} \cup S_{i_j - 1}) - v(S_{i_j - 1})}_{x_j} = x(R)$$

Core is nonempty iff the optimal value for the following linear optimation problem is $v(N)$:

$$\text{minimize}_{x} \ \overbrace{\sum_{i \in N} x_i}^{x_i(N)}$$

$$x \text{ such that } \underbrace{\sum_{i \in S} x_i}_{x(S)} \geq v(S) \ \forall S \quad \leftarrow \text{L.M. } \lambda_S.$$

Consider dual

$$\text{optimal value} = \min_{x} \max_{\lambda \geq 0} x(N) + \sum_S (v(S) - x(S))\lambda_S$$

$$= \max_{\lambda \geq 0} \min_{x} \sum_S v(S)\lambda_S + \sum_{i \in N} x_i \left[1 - \sum_{S: i \in S} \lambda_i\right]$$

$$= \max_{\lambda \geq 0} \sum_S v(S)\lambda_S$$

$$\sum_{S: i \in S} \lambda_S \leq 1 \text{ for all } i$$

λ is a balaned collection of weights

Core in not empty iff this value $= v(N)$

- Result called Bondareva-Shapley theorem

Here $\sum_S v(S)\lambda_S$ is a fractionalized version of $\max\limits_{\text{partitions } S_1, \ldots, S_k} \sum v(S_k)$

$N = \{1,2,3\}$

<u>Back to example.</u> $v(|S|) = \begin{cases} 1 & |S| = 3 \\ \alpha & |S| = 2 \\ 0 & \text{else} \end{cases}$

Need $\underbrace{\tfrac{1}{2} v(1,2) + \tfrac{1}{2} v(1,3) + \tfrac{1}{2} v(2,3)}_{} \le 1$

$\tfrac{3}{2}\alpha \le 1 \qquad \alpha \le 2/3$ ✓

So Bondareva-Shapley gives same result we observed earlier.

<u>Definition</u> $(v(S) : S \subset N)$ (with $v(\emptyset) = 0$) is <u>superadditive</u> if

$$v(S_1 \cup S_2) \geq v(S_1) + v(S_2) \text{ for all } S_1, S_2 \subset N \text{ with } S_1 \cap S_2 = \emptyset$$

(Implies cohesiveness).

Given the interpretation of a coalition game $\langle N, v \rangle$, it is reasonable to assume v is superadditive. Moreover, if v is cohesive, then the core of v and the core of the least subadditive majorant of v are the same.

If v is not cohesive, it makes sense to replace v by its least subadditive majorant.

Given a coalition game $\langle N, v\rangle$, the game (from Myerson sect. 9.3, result of Kaneko and Wooders '82) the k-fold replicated game has set of players $N' = N \times \{1, \dots, k\}$, with k players of each type

For $S \subseteq N'$ let $w(S') = w(S)$ if S' consists of exactly one player of each type in S for some $S \subseteq N$, and $w(S') = \min_{i \in N} v(\{i\})$ else. Let v be the least subadditive majorant of w.

<u>Proposition</u> (Kaneko + Wooders '82) For some integer $K \geq 1$. the core of the k-fold replicated game is not empty for every k that is a positive multiple of K.

The proof is based on the fact

$$\max_{x:\, x(S) \geq v(S)} \sum_{i \in N} x_i \;=\; \max_{\substack{\lambda \geq 0: \\ \sum_{S: i \in S} \lambda_S \leq 1 \text{ for all } i}} \sum_S v(S)\lambda_S$$

and solutions (x^*, λ^*) exist with λ_S rational.

So $\lambda_S = \frac{k_S}{K}$ for some $K \geq 1$ and $(k_S : S \subset N)$.

$\square$

<u>Example</u> Suppose $2/3 < \alpha < 1$ and consider coalition game $N = \{1, 2, 3\}$, $v(S) = \begin{cases} 1 & |S| = 3 \\ \alpha & |S| = 2 \\ 0 & \text{else.} \end{cases}$

Recall core is empty. The <u>two</u> fold replicated game have $N' = \{1, 1', 2, 2', 3, 3'\}$, $v'(N') = 3\alpha$

$$v'(N') = 3\alpha = v'(\{1, 2\}) + v'(\{1', 3\}) + v'(\{2', 3'\})$$

(<u>Note</u> $v'(N')$ is not simply $2v(N)$). Check that $\left(\frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}\right)$ is in the core of the 2-fold replicated game.

Market with transferable utilities

(Gives rise to an important family of coalitional games)

Such market is a 4-tuple $(N, \ell, (\omega_i), (f_i))$ consisting of

- finite set N of agents
- $\ell \geq 1$ number of goods
- $\omega_i \in \mathbb{R}_+^\ell$ for each i - initial endowment
- $f_i : \mathbb{R}_+^\ell \to \mathbb{R}_+$ - production function (aka utility function or happiness function) assumed increasing, continuous, concave.

Can model as a coalition game with characteristic function v defined by

$$v(S) \triangleq \max\left\{ \sum_{i \in S} f_i(z_i) : z_i \in \mathbb{R}_+^\ell, \sum_{i \in S} z_i = \sum_{i \in S} \omega_i \right\}$$

= worth of coalition i -- the maximum production achievable by coalition S on its own.

Idea is that agents can benefit through cooperation.

$\langle N, v \rangle$ is cohesive: $v(N) \geq \sum v_i(S_i)$

why? (Easy) $S_1 \cdots S_k$ any partition of N.

Maximum welfare allocation

$$\max_{z \geq 0} \sum_i f_i(z_i)$$

$\sum_i z_i = \sum w_i$ ↑ same result using $\leq$

use $p \in \mathbb{R}^l_+$ as Lagrange multiplier

Use duality again (Slater condition is satisfied if $\omega_i > 0$ coordinate wise

$$\text{optimal value} = \max_z \min_p \sum_i f_i(z_i) + \sum_i p^T(w_i - z_i)$$

$$= \min_p \sum_i \underbrace{\max_{z_i} \{f_i(z_i) - p^T(z_i - \omega_i)\}}_{\text{payoff of agent } i \text{ subject to price vector } p.}$$

Solution (p^*, z^*) is called a <u>competitive equilibrium</u> in classical economies.

- That is z^* is an allocation: $\sum_i z_i \leq \sum_i w_i$
- For each i, $z_i^* \in \arg\max_{z_i} \{f_i(z_i) - p^T(z_i - \omega_i)\}$

(or <u>Walrasian equilibrium</u>)

<u>Claim</u> The payoff vector for any competitive equilibrium is in the core (in particular, core is nonempty -- which could also be established using Bondareva-Shapley thm.)

<u>Proof</u> Need for any S,

$$\sum_{i\in S} x_i \geq v(S)$$

where

$$x_i = f_i(z_i^*) - p^T(z_i^* - \omega_i)$$

$$v(S) = \max\left\{ \sum_{i\in S} f_i(z_i) : z_i \in \mathbb{R}_+^l,\ \sum_{i\in S} z_i = \sum_{i\in S} \omega_i \right\}$$

<u>Or</u> for any S and any $(z_i : i\in S)$ with $\sum_{i\in S} z_i = \sum_{i\in S}\omega_i$,

$$\sum_{i\in S} f_i(z_i^*) - p^T(z_i^* - \omega_i) \overset{?}{\geq} \sum_{i\in S} f_i(z_i)$$

Know

$$\sum_{i\in S} f_i(z_i^*) - p^T(z_i^* - \omega_i) \geq \sum_{i\in S} f_i(z_i) \underbrace{- p^T(z_i - \omega_i)}_{\text{Sum over } i \text{ is nonnegative so}}$$

? is "yes" $\square$

Example of an Exchange economy E with transferrable payments.

$N=2$, $\ell=1$ $\omega_1=\omega_2=1$ (endowments)

\# of agents   \# commodities

$f_1(z)=\ln(z)$

$f_2(z)=4\ln(z)$

$v(\phi)=0$

$v(\{1\})=\ln(1)=0$

$v(\{2\})=4\ln(1)=0$

$$v(\{1,2\})=\max_{\substack{z_1,z_2\\ z_1+z_2\le 2}} \ln(z_1)+4\ln(z_2)$$

$\frac{1}{z_1}=\frac{2}{z_2}=p$

$\frac{2}{3}$ $\frac{16}{3}$

$$=\ln(0.4)+4\ln(1.6)=0.9637$$

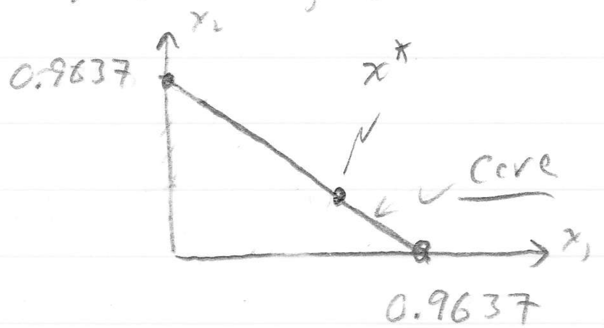
-0.9163 1.8800

Core $x_i \geq v(\{i\})=0$

$x_1+x_2=v(\{1,2\})$

$(z_1^*, z_2^*)=(\frac{2}{3},\frac{4}{3})$ is the efficient allocation

Requires transfer of 0.6 units of endowment from agent 1 to agent 2



p^* Competitive equilibrium price $f_i'(z_i)=\frac{1}{0.4}=\frac{4}{1.6}=2.5$

Under p^*, agent 1 pays $(0.6)(2.5)=1.5$

So (p^*, z^*) is the competitive equilibrium. Its payoff vector is $x^*=(\ln(0.4)+1.5,\ 4\ln(1.6)-1.5)=(0.5836, 0.3800)$

$-.9163$ 1.8800

Now consider the economy kE obtained by k-fold replication of the market E, consisting of k agents of type 1 and k agents of type 2 (This is <u>not</u> same as k fold replication of the corresponding coalitional game because coalitions with more than one agent of a given type generate worth by exchange among themselves.)

It is easy to see that the <u>competitive equilibrium</u> for kE has the same price p^* vector as for $k=1$ and allocation vector $z^{*'}=(z_1^*,\dots,z_1^*,z_2^*,\dots,z_2^*)$. So $(x_1^*,\dots,x_1^*,x_2^*,\dots x_2^*)$ is in the core of kE for all $k\geq 1$. The point (x_1^*,x_2^*) is the <u>only</u> point in the core of E with this property.

For the example, with $k=1$ (see p. C14) the point $(\bar{x}_1 = 0.1,\ \bar{x}_2 = 0.8937)$ is in the core of E.
However,

$$x' = (\underbrace{\bar{x}_1, \dots, \bar{x}_1}_{k\text{-terms}}, \underbrace{\bar{x}_2, \dots, \bar{x}_2}_{k\text{-terms}})$$ is not in the core

of the k-fold replicated market for <u>any</u> $k \geq 2$.

Indeed, let

S' = a coalition with two type 1 agents and one type 2 agent.

Then $x'(S') = 0.1 + 0.1 + 0.8937 = 1.0937$

while $$V'(S') = \max_{\substack{z_1', z_2', z_3' \\ z_1' + z_2' + z_3' = 3}} \log(z_1') + \log(z_2') + 4\log(z_3') \qquad \text{eg } (\tfrac{1}{2}, \tfrac{1}{2}, 2)$$

$$\geq \log(\tfrac{1}{2}) + \log(\tfrac{1}{2}) + 4\log(2)$$

$$= 2\log(2) = 1.3863$$

$$> x'(S') \quad \therefore x' \text{ is not in core.}$$

Basically, the type 1 agents are underpaid, so a group of them can form a coalition with a smaller number of type 2 agents to violate core constraint.

Still for the same example, let (x_1, x_2) be any point in the core of E, and for $k \ge 1$ let $x' = (\underbrace{x_1, \cdots x_1}_{k\text{ times}}, \underbrace{x_2 \cdots x_2}_{k\text{ times}})$

We'll show that if x' is in the core of E for all $k \ge 1$ then $x = x^*$, (X^* = competitive equilibrium payoff vector)
Let $v(i,j)$ be the worth of a coalition of i type 1 agents and j type 2 agents, $i+j \ge 1$

$$v(i,j) = \max_{\substack{z'_1, \cdots z'_{i+j} \\ \sum_{a=1}^{i+j} z'_a \le i+j}} f_1(z'_1) + \cdots + f_i(z'_i) + f_2(z'_{i+1}) + \cdots + f_2(z'_{i+j})$$

Solution $z'_1 = \cdots = z'_i = z_1$
$z'_{i+1} = \cdots = z'_{i+j} = z_2$

where $f'_1(z_1) = f'_2(z_2) = p(\theta)$

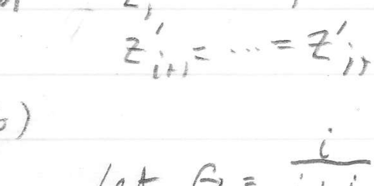
Let $\theta = \frac{i}{i+j}$ = fraction of type 1 agents

$$= (i+j) \cdot \overbrace{\max_{z:\ \theta z_1 + (1-\theta) z_2 = 1} \theta f_1(z_1) + (1-\theta) f_2(\theta)}^{\text{call this } \varphi(\theta)}$$

$$= (i+j)\, \varphi(\theta)$$

For $f_1(z_1) = \ln(z_1)$ and $f_2(z_2) = 4\ln(z_2)$ we find

$z_1(\theta) = \frac{1}{\theta + 4(1-\theta)}$ $\quad z_2(\theta) = 4 z_1(\theta)$



(The value of the corresponding Lagrange multiplier is
$p(\theta) = f'_1(z_1(\theta)) = f'_2(z_2(\theta)) = 4 - 3\theta$
Also, $\varphi'(\theta) = x_1(\theta) - x_2(\theta)$ where $x_i(\theta) = f_i(z_i(\theta)) - p(\theta)(z_i(\theta) - w_i)$
(See next page. $\theta = \frac{1}{2}$ corresponds to welfare maximization for the entire market, $i = j = k$.

C18

(Proof that $\varphi'(\theta) = x_1(\theta) - x_2(\theta)$)

(Use $z_1 = a$, $z_2 = b$)

$$\varphi(\theta) = \max_{\substack{z=(z_1, z_2) \geq 0 \\ \theta z_1 + (1-\theta) z_2 \leq 1}} \{ \theta f_1(z_1) + (1-\theta) f_2(z_2) \}$$

$$= \max_{z \geq 0} \min_{p \geq 0} \theta f_1(z_1) + (1-\theta) f_2(z_2) + p(\overset{\theta + (1-\theta)}{\overset{\downarrow}{1}} - \theta z_1 - (1-\theta) z_2)$$

$$= \min_{p \geq 0} \Big[\theta \underbrace{\max_{z_1} \{ f_1(z_1) - p(z_1 - 1) \}}_{x_1(\theta) \text{ (or } x_1(p(\theta)))} + (1-\theta) \underbrace{\max_{z_2} \{ f_2(z_2) - p(z_2 - 1) \}}_{x_2(\theta)} \Big]$$

The solution $(p(\theta), z(\theta))$ is the competitive equilibrium for a market with fraction θ of type 1 agents and fraction $(1-\theta)$ of type 2 agents. Let $(x_1(\theta), x_2(\theta))$ be the corresponding payoff vector.

$$\varphi(\theta) = \theta x_1(\theta) + (1-\theta) x_2(\theta) \qquad \left(= \frac{x_1^* + x_2^*}{2} \text{ at } \theta = \tfrac{1}{2} \right)$$

$$\varphi'(\theta) = x_1(\theta) - x_2(\theta) + \theta x_1'(\theta) + (1-\theta) x_2'(\theta)$$

$$= x_1(\theta) - x_2(\theta) + \underbrace{\left(\theta \frac{dx_1}{dp} + (1-\theta) \frac{dx_2}{dp} \right)}_{\text{zero at minimizer, } p(\theta)} p'(\theta)$$

$$\boxed{\varphi'(\theta) = x_1(\theta) - x_2(\theta)}$$

(Above, x_i is a function of p and p is a function of θ, so indirectly x_i is a function of θ, same for x_2.)

Suppose $(x_1, \cdots, x_1, x_2, \cdots x_2)$ is in the core of kE. Then

$$\begin{array}{l} k x_1 + k x_2 = v(k,k) \\ \underline{(k-1) x_1 + k x_2 \geq v(k-1, k)} \end{array}$$

so $$x_1 \leq v(k,k) - v(k-1,k)$$

$$= 2k\,\varphi(\tfrac{1}{2}) - (2k-1)\,\varphi\left(\tfrac{k-1}{2k-1}\right)$$

$$= 2k\,\varphi(\tfrac{1}{2}) - (2k-1)\left(\varphi(\tfrac{1}{2}) + \varphi'(\tfrac{1}{2})\left(-\tfrac{1}{2(2k-1)}\right) + O\left(\tfrac{1}{k^2}\right)\right)$$

$$\frac{k-1}{2k-1} - \frac{1}{2} = \frac{2(k-1)-(2k-1)}{2(2k-1)} = -\frac{1}{2(2k-1)}$$

$$= \varphi(\tfrac{1}{2}) + \tfrac{1}{2}\varphi'(\tfrac{1}{2}) + o(\tfrac{1}{k})$$

$$= \frac{x_1^* + x_2^*}{2} + \frac{x_1^* - x_2^*}{2} + o(\tfrac{1}{k})$$

$$= x_1^* + o(\tfrac{1}{k})$$

Similarly, $x_2 \leq x_2^* + o(\frac{1}{k})$. know $x_1 + x_2 = x_1^* + x_2^*$

So $x_i = x_i^* + o(\frac{1}{k})$ for $i = 1, 2$.

$\therefore$ If $(\underbrace{x_1 \cdots, x_1}_{k \text{ times}}, \underbrace{x_2, \cdots x_2}_{k \text{ times}})$ is in the core of kE for all $k \geq 1$, then $x_1 = x_1^*$, $x_2 = x_2^*$ (in this sense, the core shrinks to the payoff vector the the competitive equilibrium of E as $k \to \infty$.) $\square$

The Shaply value

Return to the setting of a general coalitional game $\langle N, v\rangle$. An alternative to considering the core is to somehow assign a feasible payoff vector $(x_i : i \in N)$ to any such game. (Feasible means $\sum_{i \in N} x_i = v(N)$). The Shapley value can be given as follows.

Let $\mathcal{P}_n$ denote the set of $n!$ orderings of $N = \{1, \cdots, n\}$ (aka permutations)

For $R \in \mathcal{P}_n$, let $S_i(R) = \{j \in N : j \prec i\}$ (not including i)

$\Delta_i(S) = v(S \cup \{i\}) - v(S)$ for a coalition with $i \notin S$. So $\Delta_i(S)$ is the = marginal worth of i in the coalition $S \cup \{i\}$

[Shapley value definition
$$x_i \triangleq \frac{1}{n!} \sum_{R \in \mathcal{P}_n} \Delta_i(S_i(R))$$
]

(See Chap. 14 of O&R for other allocation notions for coalition games.)

Note If $\langle N, v\rangle$ is convex ($v(S_1 \cup S_2) + v(S_1 \cap S_2) \geq v(S_1) + v(S_2)$)

then $(x_i) = (\Delta_i(S(R)))_{i \in N}$

(as we saw earlier) is in the core of $\langle N, v\rangle$ for any R. Since the core of $\langle N, v\rangle$ is convex, the Shapley value is in the core for any convex coalitional game with transfers.

Shapley showed the mapping

$$v \xrightarrow{F} x^{shapley} \quad (\overset{N}{\text{fixed}})$$

is the unique one subject to three axioms for a given set N.

- linear: $F(v_1 + v_2) = F(v_1) + F(v_2)$
- symmetric: with respect to permutations of agents
- no value to dummies: $F_i(v) = 0$ if $v(S) = v(S \cup \{i\})$ for every coalition S with $i \notin S$.

(see O+R for proof)

Example

Consider a majority coalition formed in a parliment, consisting of three blocks with 20, 30, 50 seats repectively. Suppose the threshold for a majority coalition is 65 seats. Thus, if the blocks are s, m, l respectively, $v(\{s,l\}) = v(\{m,l\}) = v(\{s,m,l\}) = 1$.

To compute Shapely value

R	$(\Delta_i(S_i(R))$
s m l |	(0, 0, 1)
s l | m	"
m s l |	"
m l | s	"
l s | m	(1, 0, 0)
l m | s	(0, 1, 0)

$$x_{shaply} = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{2}{3}\right)$$

$$\text{Core} = \{(0,0,1)\}$$

— $(0,0,1)$ is in core because any winning coalition includes l. If $x_1 > 0$ then $x_2 + x_3 < v(\{2,3\}) = 1$ so $x_1 = 0$ for vectors in core. $x_2 = 0$ as well.