OM-1 Lecture 21 (continued)

<u>(Revenue) Optimal Mechanisms</u> (Myerson '82, presentation following Krishna Chap. 5.)

Seller has a single object to sell

Buyers have independent private values $(X_i)_{i \in I}$

$X_i \in [0, \omega_i]$, pdf f_i, CDF F_i

A selling <u>mechanism</u> is a triple $(\mathbb{B}, \pi, \mu)$

$\mathbb{B}$ = space of bid vectors

$\pi : \mathbb{B} \to \Delta$ - probabilistic allocation rule

probability vector over set of buyers

$\mu : \mathbb{B} \to \mathbb{R}^N$ - payment rule

A mechanism induces a game of <u>incomplete information</u> among the buyers, where the type of buyer i is his value X_i for the object.

Following Myerson '82, we seek the mechanism such that for a specified Bayesian equilibrium, the expected revenue is maximized.

Revelation Principle

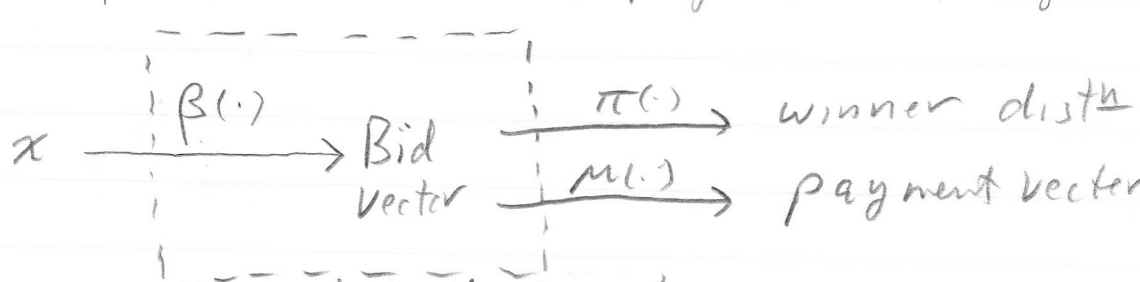
A direct mechanism (Q, M) consists of

$Q : \mathcal{X} \to \Delta$ — random allocation ($\mathcal{X}$ ← space of $(x_1, \ldots, x_N)$, value vectors)

$\underline{M} : \mathcal{X} \to \mathbb{R}^N$ — payment rule

Proposition Given a mechanism $(\underline{\mathcal{B}}, \pi, \mu)$ and an equilibrium β for that mechanism, there exists a direct mechanism such that reporting truthfully is an equilibrium, with the same outcomes as μ for every $\underline{x}$. (i.e. same dist$^{\underline{n}}$ of winner and same payoffs.)

Proof $\beta_i(x_i)$ is response of player i for the equilibrium



Equivalent direct mechanism

$Q(x) = \pi(\beta(x)) \qquad M(x) = \mu(\beta(x))$ $\square$

Incentive Compatibility (IC)

If buyer i bids z_i, she is allocated the object with probability

- $q_i(z_i) = \int_{X_{-i}} Q_i(z_i, x_{-i}) f_{-i}(x_{-i})\, dx_{-i}$

and she expects to pay:

- $m_i(z_i) = \int_{X_{-i}} M_i(z_i, x_{-i}) f_{-i}(x_{-i})\, dx_{-i}$

Incentive compatibility for buyer i is determined entirely by the functions q_i, m_i over the range of possible values $x_i \in [0, \omega_i]$:

IC: $U(x_i) \triangleq q_i(x_i) x_i - m_i(x_i) \geq q_i(z_i) x_i - m_i(z_i)$

(Note: constraints are linear.)

For $x < y$,

$$q(x)\,x - \cancel{m(x)} \geq q(y)\,x - \cancel{m(y)}$$

$$+ \quad q(y)\,y - \cancel{m(y)} \geq q(x)\,y - \cancel{m(x)}$$

$$(q(x) - q(y))(x - y) \geq 0 \quad \therefore \ \underline{q \text{ nondecreasing}}$$

IC implies (recall $U_i(x_i) = q_i(x_i)\,x_i - m_i(x_i)$)

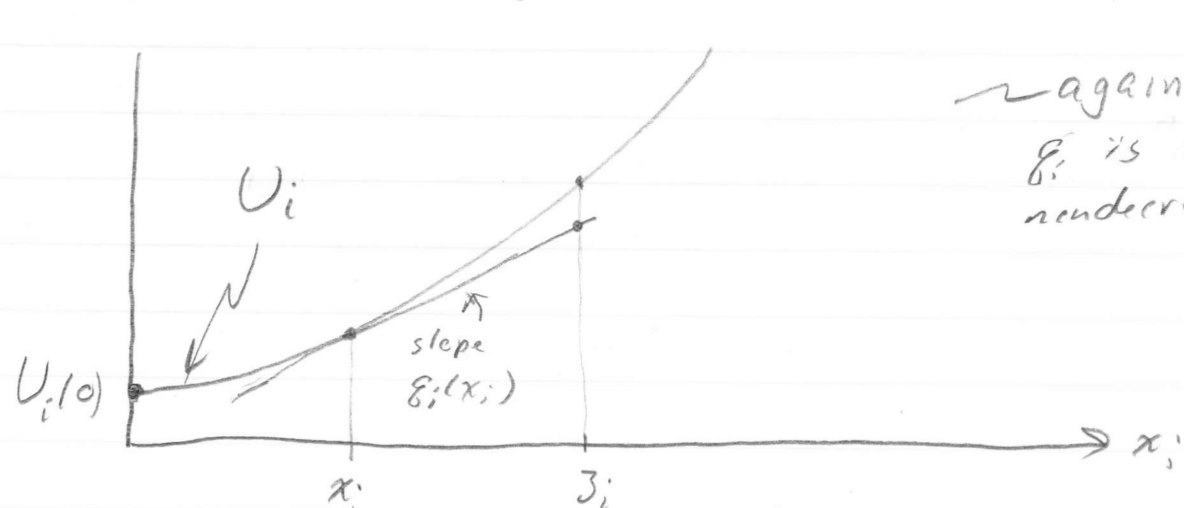
$$U_i(x_i) = \max_{z_i} \ q_i(z_i)\,x_i - m_i(x_i)$$ ← U_i is convex

$\therefore U_i$ is absolutely continuous [so differentiable a.e. and integral of its derivative.

For $x_i, z_i \in [0, \omega_i]$

$$U_i(z_i) \geq q_i(x_i)\,z_i - m_i(x_i) = U_i(x_i) + q_i(x_i)(z_i - x_i)$$

i.e. $q_i(x_i)$ is a subgradient of U_i at x_i



← again see q_i is nondecreasing.

$\therefore U_i$ is completely determined by $U_i(0)$ and q_i.

$$\underbrace{q_i(x_i)\,x_i - m_i(x_i)}_{U_i(x_i)} = U_i(0) + \int_0^{x_i} q_i(t_i)\,dt_i$$

$$\boxed{m_i(x_i) = \underbrace{m_i(0)}_{-U_i(0)} + q_i(x_i)\,x_i - \int_0^{x_i} q_i(t_i)\,dt_i \quad (*)}$$ Revenue Equivalence

← does not involve f_i

So if the direct mechanism (Q, M) is IC then q is nondecreasing and m is determined by $(*)$.

Conversely, if q is nondecreasing and m is determined by $(*)$ then (Q, M) is IC. (for converse, follow steps in reverse order)

(If q_i is differentiable, $m_i'(x_i) = x_i q_i'(x_i)$ -- makes sense.)

Individual Rationality (IR)

Def: Mechanism is IR if for all i, x_i, $U_i(x_i) \geq 0$.

IR = player is not forced to have negative payoff (a reasonable property for revenue maximization problem -- else seller can just set arbitrarily high payments).

$$IR \iff \underbrace{U_i(0)}_{= q_i(0)\cdot 0 - m_i(0)} \geq 0 \iff m_i(0) \leq 0$$

Illustrating revenue equivalence

Example Suppose $q_i(x_i) = x_i$ for $0 \le x_i \le 1$.

What is the maximum payment function m_i satisfying IR and IC constraints?

$m_i' = q_i' x_i$ Take $m_i(0) = 0$ to maximize revenue

$m_i' = x_i$

$$\boxed{m_i(x_i) = \tfrac{1}{2} x_i^2}$$

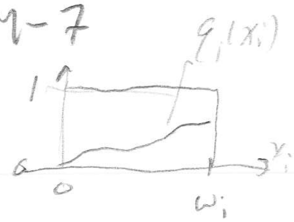
Double Check If your true value is x_i and you report z_i, your expected payoff is

$x_i q_i(z_i) - m_i(z_i)$ or $x_i z_i - \frac{1}{2} z_i^2$.

This is maximized by $z_i = x_i$ ✓.

Or $q_i(x_i) = 1 - e^{-x_i}$ $x_i \ge 0$ $\iff$ $m_i' = x_i q' = x_i e^{-x_i}$

$m_i(x_i) = 1 - (1 + x_i) e^{-x_i}$



Note If player i were playing a second price auction and q_i were the CDF of the highest bid W of the other players, then for bid x_i, player i would get item with probability $q_i(x_i)$, and mean the payment would be

$$m_i(x_i) = q_i(x_i) \overbrace{E[\,W \mid W \le x_i\,]}^{\text{conditional payment}}$$

W has CDF q_i

Conditional pdf $q_i'(t)/q_i(x_i)$

$$= \cancel{q_i(x_i)}\left(\int_0^{x_i} t\, q_i'(t)\,dt \,/\, \cancel{q_i(x_i)}\right)$$

Integrate by parts

$$\begin{array}{cc} t & q' \\ 1 & q_i \end{array}$$

$$= x_i q_i(x_i) - \int_0^{x_i} q_i(t)\,dt$$

Same as formula derived for m_i earlier. As expected, given mean payment is determined by function q_i and $m_i(0)$.

Using the revelation principle and the revenue equivalence property of IC mechanisms, we now address (revenue) <u>optimal mechanisms</u>.

Given $(f_i)_{1\le i\le N}$ where f_i is a pdf on $[0, w_i]$

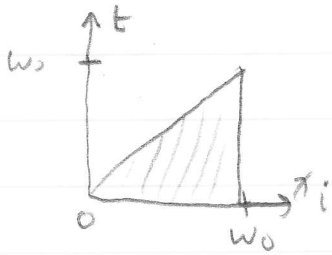
$$\text{maximize} \quad E\Big[\sum_i m_i(X_i)\Big]$$

over (Q, M)

subject to IC, IR

Take $m_i(0)=0$ (optimal) so $m_i(x) = q_i(x_i)x_i - \int_0^{x_i} q_i(t)\,dt$

$$E[m(X_i)] = \int_0^{w_i} q_i(x_i)\,x_i\,f_i(x_i)\,dx_i$$



$$- \int_0^{w_0}\left(\int_0^{x_i} q_i(t)\,dt\right) f(x_i)\,dx_i$$

2^{nd} term

Key Step!

$$= -\int_0^{w_0}\Big(\underbrace{\int_t^{w_0} f(x_i)\,dx_i}_{1-F_i(t)}\Big)\, q_i(t)\,dt = -\int_0^{w_0} \frac{1-F(x_i)}{f(x_i)}\, f(x_i)\, q_i(x_i)\,dx_i$$

so

$$E(m(X_i)] = \int_0^{w_i} \psi_i(x_i)\,q_i(x_i)\,f_i(x_i)\,dx_i = E[\psi_i(X_i)\,I_{\{i \text{ wins}\}}]$$

where $\psi_i(x_i) = x_i - \dfrac{1-F_i(x_i)}{f_i(x_i)}$ "win" means gets object.

$\psi(X_i)$ is called the <u>virtual</u> valuation for buyer i.

So (revenue) optimal mechanism solves:

$$\max \; E\Big[\sum_{i=1}^{N} \psi_i(X_i) I_{\{i \text{ wins}\}}\Big] \qquad (**)$$

over Q subject to

$q_i(x_i)$ nondecreasing for each i.

(Recall, Q is winner selection function)

<u>If</u> $\psi_i(x_i)$ is <u>increasing</u> in x_i for each i

(i.e. the design problem is <u>regular</u>) then

the optimal assignment rule is

|| Select winner from $\arg\max_i \psi_i(X_i)$ if $\max_i \psi(X_i) \geq 0$

|| Select no winner if $\psi_i(X_i) < 0$ for all i.

Indeed, this rule maximizes $(**)$ over all assignment rules (ignoring the constraint that $q_i(x_i)$ be nondecreasing) but the constraint is satisfied anyway. The revenue is

$$E[\max\{\psi_1(X_1), \cdots, \psi_N(X_N), 0\}]$$

Payment rule for optimal auction: Need

$$m_i(x_i) = q_i(x_i)\,x_i - \int_0^{x_i} q_i(t)\,dt \quad \text{ⓐ}$$

This only specifies average payment, and is paid whether player i wins or not.

Any choice of $M_i(x)$ with $E[M_i(X) \mid X_i = x_i] = m_i(x_i)$ would be revenue optimal subject to IC, IR constraints. Think of second price auction can help us determine good choice for M_i.

Let $y_i(x_{-i}) = \min\{z_i : \psi_i(z_i) \geq 0 \text{ and } \psi_i(z_i) \geq \psi_j(x_j)\ j \neq i\}$

$=$ minimum value i would need to win, given x_{-i}.

Then let $M_i(x) = y_i(x_{-i})\, I_{\{i \text{ wins}\}}$

We can argue that as in a second price auction, this is the most buyer can charge subject to IC.

As a double check, we check $E[M_i(X) \mid X_i = x_i] \overset{?}{=} m_i(x_i)$

Check

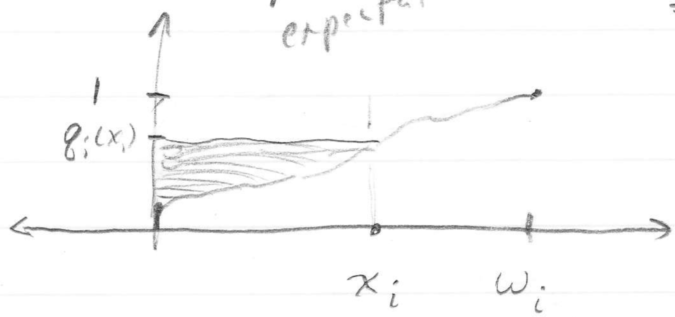
$$E[M_i(X) \mid X_i = x_i] = E\left[y_i(X_{-i})\, I_{\{y_i(X_{-i}) < x_i\}}\right]$$

Integration by parts, or area rule for expectations

$$= \int_0^{x_i} q_i(x_i) - q_i(t)\, dt$$

$$= x_i q_i(x_i) - \int_0^{x_i} q_i(t)\, dt$$

$$= m_i(x_i) \quad \checkmark$$



CDF of $y_i(X_{-i})$ is $q_i(t)$

Note The reserve value for buyer i is given by $r_i = \inf_{z_i} \{z_i : \psi(z_i) \geq 0\}$.

Buyer i can't win if bid is less than r_i.

If $X_j < r_j$ for all buyers, seller doesn't allocate. It is assumed seller has free disposal – i.e. no cost to not allocate to a buyer.

Recovery of 2^{nd} Price Auction

$$\max E[X_i I_{\{i \text{ is winner}\}}]$$

over (Q, M)

subject to IC, IR

Clearly the selection rule is to select

$i^* \in \arg\max X_i$. If $W_i = \max_{j: j \neq i} X_j$ then

$q_i(x_i) = P\{W_i < x_i\} = \prod_{j \neq i} F_j(x_i)$ $\quad$ q_i is CDF of W_i

q_i is nondecreasing ✓

Expect $M_i(x) = W_i I_{\{W_i < x_i\}}$ $\quad$ (2^{nd} price)

Check

$$E[M_i(X) \mid X_i = x_i] = E[W_i I_{\{W_i < x_i\}}]$$

$$= x_i q_i(x_i) - \int_0^{\overset{\text{omega}}{\omega_i}} q_i(t)\,dt$$

as in revenue optimal case.

Auctions with Interdependent Values
(Milgrom + Weber 1983 paper is key,
Presentation here based on Krishna, Chap. 6)

Focus on Bayes-Nash equilibrium for basic auctions.

First look at

Independent Private Values

$X_1, \cdots X_n$ independent, $X_i \sim f_i$

We've discussed 2nd price auction -- weakly dominant strategy is truthful bidding.

English auction - Seller price increases continuously, starting at zero. Buyers decide when to drop out and see each other drop out. At the time only one buyer remains, that buyer wins item and pays the price at that time.

Does a player in English auction have a

dominant strategy? ... Yes, drop when price equals value. Equivalent to second price auction. Even though a buyer i with value X_i sees some others dropping out at prices below X_i, that information causes no need to deviate.

<u>First price</u> (Symmetric case) Let $Y_1 = \max\{X_2, \dots, X_N\}$. Suppose other buyers use increasing strategy β. Conditional payoff of buyer 1 given $X_1 = x$ and bid b is

$$\pi_1(b, x) = P[\beta(Y_1) < b](x-b)$$

$$= P[Y_1 \le \beta^{-1}(b)](x-b)$$

$$= G(\beta^{-1}(b))(x-b) \qquad G = \text{CDF of } Y_1 = F^{n-1}$$

MW-3

$$\frac{d}{db}(\;) = \frac{G'(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x-b) - G(\beta^{-1}(b)) = 0$$

If player 1 also uses β, then $\beta^{-1}(b) = x$.

Yields

$$G'(x)(x-\beta(x)) - G(x)\beta'(x) = 0$$

$$\underbrace{G'(x)}_{g(x)}x = (G(x)\beta(x))' \qquad g = G' = \text{density}$$

$$G(x)\beta(x) = \int_0^x y\,g(y)\,dy$$

$$\beta(x) = \frac{1}{G(x)}\int_0^x y\,g(y)\,dy$$

$$\boxed{\beta^{I}(x) = E[Y_1 \mid Y_1 < x]}$$

This was derived as necessary condition for symmetric Bayes NE. Can show sufficient (Krishna sect. 2.3)

English

Revenue Ranking $\quad R^{II} = R^{E} \geq R^{I}$

(symmetric, independent private valuation.

(we saw 2nd price is revenue optimal.

- In many situations buyers don't know their values for the object at auction time. Perhaps have signals X_i. One model:

$$V_i = g_i(X_1, \cdots X_n) \quad \text{typically } g \text{ nondecreasing.}$$

(could be $E[V_i | X_1 \cdots X_n]$)

Moreover, the signals may be dependent.

- What are Bayes-Nash equilibria like for the three auctions?
- Are second price and English auctions still equivalent. (No -- why? More info.)
- What happens to revenue ranking?

Milgrom + Weber (1983) addresses these questions. Assume symmetry, examine symmetric equilibria, assume a strong version of positive correlation of signals.

<u>Def</u> $X_1, \cdots X_n$ with joint pdf over $\prod_{i=1}^{N} [0, w_i]$

are <u>affiliated</u> (M+W notation, others say ———)

if $f(x \wedge y) f(y \vee x) \geq f(x) f(y)$

(i.e. $\log(f(x,y))$ is supermodular:

$$\log f(x \wedge y) + \log f(x \vee y) \geq \log f(x) + \log f(y)$$

(This terminology of Milgrom and Weber has stuck within economics literature. Know to statististicians as total positivity.

[We'll not need/use it, but a consequence is the FKG inequality: $Cov(g(X), h(X)) \geq 0$ for any nondecreasing functions g and h.]

(Follows from Ahlswede-Daykin inequality which has a simple proof by induction in discrete setting. due to Bollman (1980). See Ahlswede and Blinovski, Lectures on Adv in Combinatorics, Springer, 2008)

Sometimes called association property.

Some consequences of affiliation

① (Equivalent to affiliation) For any subset I of the coordinates, the conditional pdf of X_{-I} given X_I is monotone nondecreasing from $(\mathbb{R}^I$, pointwise order$)$ to pdf's for X_{-I} in likelihood ratio order.

e.g. $I = \{k+1, \cdots, n\}$: $\dfrac{f(x_1, \cdots, x_k \mid x_{k+1} \cdots x_n)}{f(x_1, \cdots x_k \mid x'_{k+1}, \cdots, x'_n)}$

is weakly increasing in $(x_1, \cdots, x_k)$ if $(x_{k+1}, \cdots x_n) \overset{\text{pointwise}}{\geq} (x'_{k+1}, \cdots, x'_n)$

② For any (bounded) non increasing function γ on $\mathbb{R}^N$,

$$E[\, \gamma(X_1, \cdots, X_n) \mid X_1 \in [a_1, b_1], \cdots, X_n \in [a_n, b_n] \,]$$

is weakly increasing in $(a_1, \cdots, a_n, b_1, \cdots, b_n)$

③ $X_1, Y_1, \cdots, Y_{N-1}$ is affiliated, where $Y_1, \cdots, Y_{N-1}$ is the vector of order statistics for $X_2, \cdots, X_n$.

MW-7

4/18/13
Lecture 24

[2^{nd} Price Auction
Symmetric model]

- Joint Dist^{n} of $X_1, \cdots X_N$ symmetric

- $V_i(x) = u(x_i, x_{-i})$ and $u_i(x_i, \bullet)$ is
no i
symmetric in coordinates of x_{-i},
and $u(x)$ is nondecreasing in x.

$V_i = v_i(X) = u(X_i, X_{-i})$

Define $v(x,y) = E[V_1 \mid X_1 = x, Y_1 = y]$
$= E[u(x, y, Y_2, \cdots, Y_{N-1}) \mid X_1 = x, Y_1 = y]$

Assume $v(x,y)$ is nondecresing in x, y

and strictly increasing in x. (Follows, for example, if $u(x_i, x_{-i})$ is strictly increasing in x_i and X's are affiliated.)

For 2^{nd} price auction, symmetric joint distribution of signals and payoffs

Proposition Let $\beta^{II}(x) = v(x,x)$. Then

$(\beta^{II}, \cdots, \beta^{II})$ is a symmetric Bayes-Nash equilibrium (can show uniqueness too -- no other symmetric equilibrium -- we skip.)

<u>Proof</u> We check first that $(\beta^{II}, \dots, \beta^{II})$ is a Bayes-Nash equilibrium. By symmetry we can focus on buyer 1. Suppose buyers 2 through N use β^{II}, and buyer 1 observes $X_1 = x$. What bid b is the best response for buyer 1? Since auction is second price, buyer wins V_1 if $v(Y_1, Y_1) < b$. Suppose buyer one <u>knew</u> the value of Y_1. He would then calculate his conditional expected payoff as

$$\underbrace{\Big[\overset{E[V_1 \mid X_1 = x, Y_1]}{\overbrace{v(x, Y_1)}} - \overset{\text{payment}}{\overbrace{v(Y_1, Y_1)}}\Big]}_{\geq 0 \iff x \geq Y_1} I_{\{v(Y_1, Y_1) \leq b\}}$$

So buyer 1 would be best off winning if $x \geq Y_1$ and losing else. That is (uniquely) accomplished by letting $b = v(x, x) = \beta^{II}(x)$. $\square$

English auction - strategies

A strategy for a bidder in an English auction is an $(N-1)$-tuple of functions $\beta = (\beta^N, \dots, \beta^2)$ where $\beta^k(x, p_{k+1}, \dots, p_N)$ is the price at which the bidder with signal x would drop out if k bidder's remain and the other $N-k$ bidders dropped out at prices $p_{k+1} \le \dots \le p_N$.

If the strategies of the other bidder are known, and if the β^k function of each bidder is a strictly increasing function of the bidders value x_j, then a bidder can deduce the values of these bidders that dropped out.

Assume a symmetric dependence of values on signals: $v_i(x) = u(x_i, x_{-i})$, where $u(x_i, x_{-i})$ is a symmetric function of x_{-i} for x_i fixed. Also, assume $u(x_i, x_{-i})$ is nondecreasing in x_{-i} and strictly increasing in x_i.

Given u, determine a strategy β recursively as follows:

Determined by $\beta^N(x_N) = p_N$

$$\beta^N(x) = u(x, \dots, x)$$

Determined by $\beta^{N-1}(x_{N-1}, p_N) = p_{N-1}$

$$\beta^{N-1}(x, p_N) = u(x, \dots, x, x_N)$$

$$\beta^{N-1}(x, p_{N-1}, p_N) = u(x, \dots, x, x_{N-1}, x_N)$$

$$\vdots$$

Determined by $\beta^3(x_3, p_4, \dots, p_N) = p_3$

$$\beta^2(x, p_3, \dots, p_N) = u(x, x, x_3, \dots, x_N)$$

<u>Note</u> - Does not involve joint pdf f.

- Choice of β^k implicity assumes other bidders used same $\beta^{k+1}, \dots, \beta^N$. i.e. symmetric strategies.

<u>Proposition</u> The symmetric strategy profile of all bidders using β (determined by function w) is a Bayes-Nash equilibrium. (even if joint distribution of the signals $X_1, \cdots, X_n$ is not symmetric.)

<u>Proof</u> Focus on bidder 1, assuming all the other bidders follow β. Let $y_1, \cdots, y_{N-1}$ be a realization of the order statistics $Y_1, \cdots, Y_{N-1}$ based on $X_2, \cdots, X_N$, as before. Let x denote the value of X_1. Bidder 1 either stays in the auction til he wins, or he drops out earlier. If he stays in his price is $u(y_1, y_1, y_2, \cdots, y_{N-1})$ and his payoff is thus $u(x, y_1, \cdots, y_{N-1}) - u(y_1, y_1, \cdots, y_{N-1})$. It's positive if $x > y_1$ and negative if $x < y_1$. Thus, if bidder 1 follows β, he will win when payoff is positive (i.e. when $x > y_1$) and lose when payoff is negative (i.e. when $x < y_1$), as desired. $\square$

Q. For a given realization of $x_1, \cdots, x_N$, does a bidder with the largest signal x_i win?

A. Yes. To see this, note that the functions $\beta^k(x, p_{k+1}, \cdots, p_N) = u(x, \cdots, x, x_{k+1}, \cdots, x_N)$ are all increasing in x.

Q. Ultimately, the value of the object to buyer i is $V_i(x) = u_i(x_i, x_{-i})$. Does a buyer with largest value always win?

A. No, not always. For example, if

$$V_i(x) = u(x_i, x_{-i}) = x_i + 2\sum_{j: j \neq i} x_j = 2\left(\sum_{j=1}^{n} x_j\right) - x_i$$

then the bidders with largest values have the smallest signals, and are first to drop out.

Q. Bidders drop out before they know their values. Is it possible the selling price is smaller than the value of a losing bidder?

A. Yes $N=2$ $v_1(x) = x_1 + 2x_2$
$v_2(x) = x_2 + 2x_1$

$\beta^2(x) = u(x,x) = 3x$

If $x_1 < x_2$ bidder 2 wins and pays $3x_1$ and has value $x_2 + 2x_1$. Bidder 1 has an even larger value, $x_1 + 2x_2$, and lost.

Q. Would a bidder ever regret not bidding higher?

A. No. In the example just given, if bidder 2 decided to win the auction, she'd have to pay $3x_2$.
In general a losing buyer (say buyer 1, means $x_1 < y_1$) that deviates to win receives
payoff $u(x_1, y_1, \cdots, y_{N-1}) - u(y_1, y_1, \cdots, y_{N-1}) < 0$

Revenue Comparison (symmetric case)

$$R^{II} \overset{\text{second price}}{=} E[v(Y_1, Y_1) \mid Y_1 < X_1]$$

$$R^{ENG} = E[u(Y_1, Y_1, Y_2, \cdots, Y_{N-1}) \mid Y_1 < X_1]$$

$$v(y,y) = E(\overbrace{u(X_1, Y_1, Y_2, \cdots Y_{N-1})}^{V_1} \mid X_1 = y, Y_1 = y]$$

$$= E[u(Y_1, Y_1, \cdots, Y_{N-1}) \mid X_1 = y, Y_1 = y]$$

$$\overset{(*)}{\leq} E[u(Y_1, Y_1, \cdots, Y_{N-1}) \mid X_1 = x, Y_1 = y] \text{ for any } x \geq y$$

Plugging in v in expression for R^{II} and comparing to R^{ENG} yields:

$$\boxed{R^{II} \leq R^{ENG}}$$

Note There are all equalities except at (*).

Increasing the conditioning value for X_1 causes (likelihood ratio ordering) increase in $(Y_2, \cdots, Y_{N-1})$. These stochastically larger values are observed/plugged in, in case of English auctions, essentially letting seller capture the gain.