Lecture 14
3/7/13

Extensive Form Games

Nim

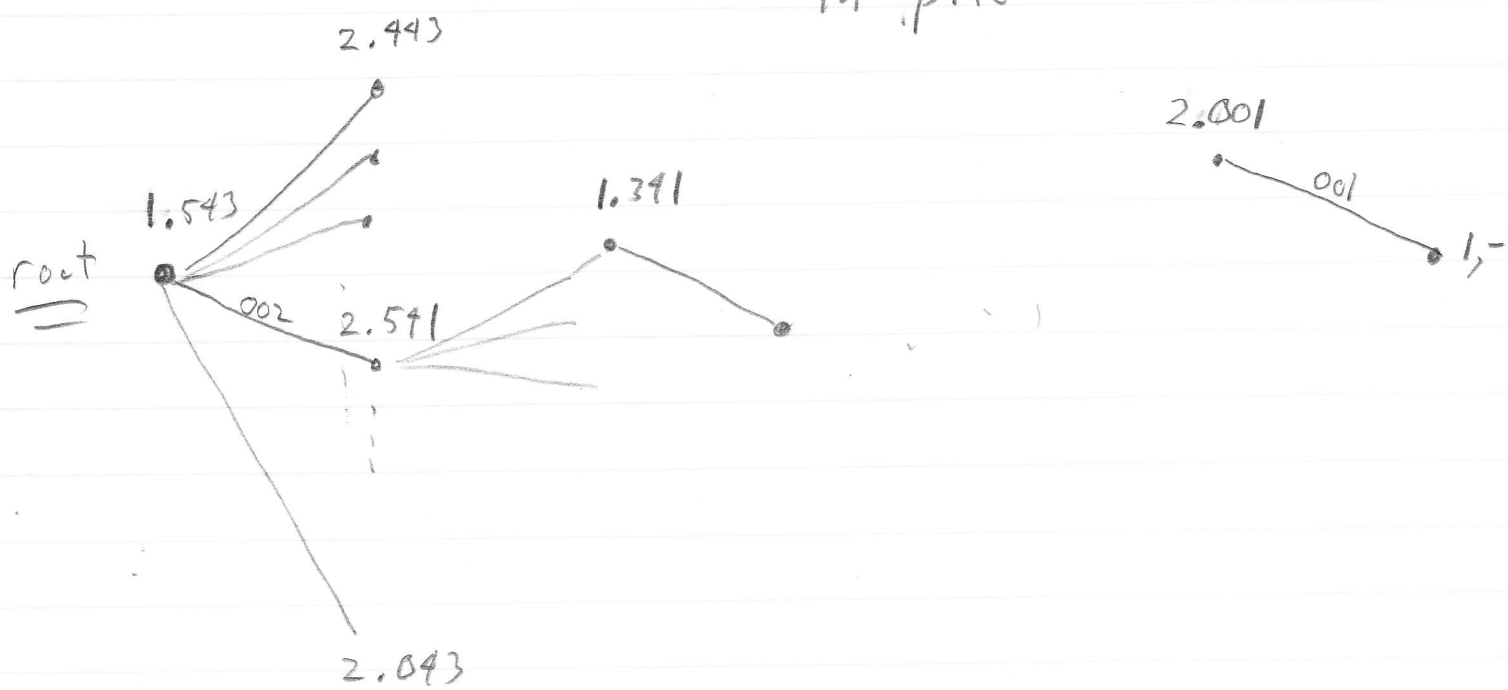
||||| |||| |||

start with three piles, sizes 5, 4, 3
play alternates between players
during turn, player removes one or more sticks from one pile.
player to remove last stick loses

- demo a few times

- how to model?

A state $r = (r_1, r_2, r_3)$ r_i = # sticks remaining in pile i



Label nonterminal nodes by player.infostate
terminal nodes by payoff vectors
edges ("branches") by actions

Nim has pefect information for each player.
Same as chess or checkers.
In principle, can solve such games by backwards induction.
(or permutations of states shown)

Winning states	Losing states
	100
(2+)00 , (1+)10	
	111 220
(2+)11 (3+)20	
	321 330
32(2+) (3+)31 (4+)21 (4+)30 33(1+)	
	440
540 44(1+)	
	541
543	

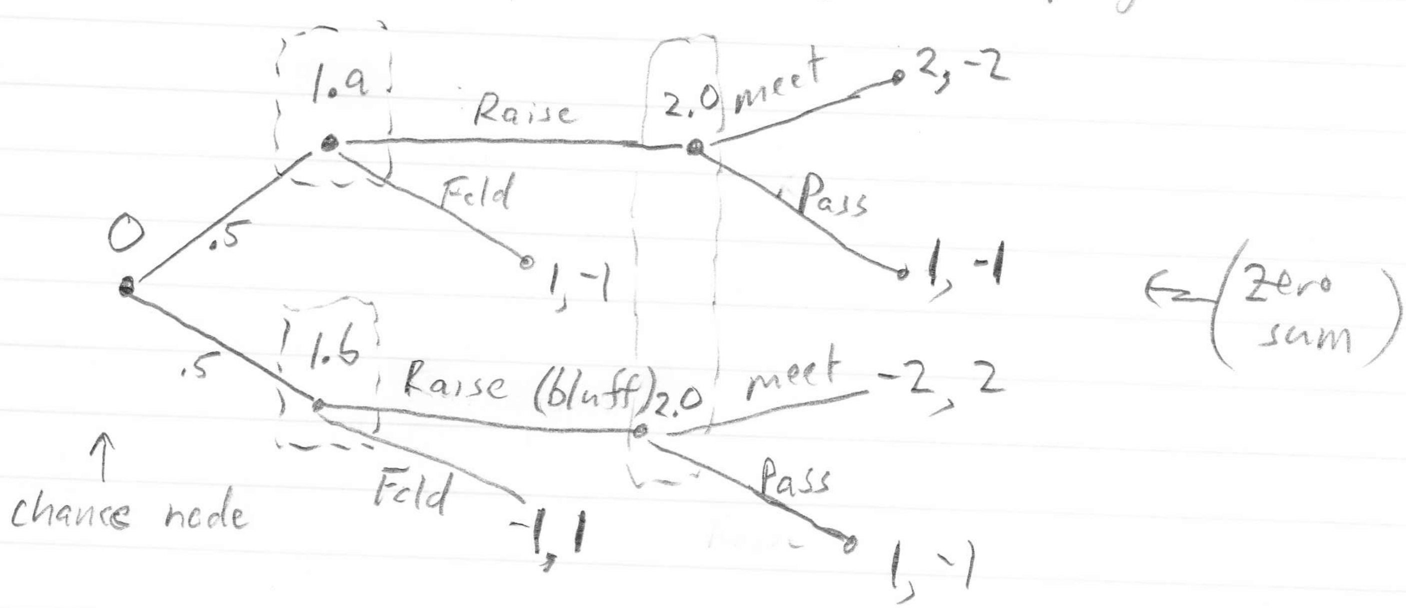
∴ Player 1 can always win the 543 game.

Theory of nim sums gives general solutions. Decompose each pile uniquely into subpiles with powers of two sticks and cancel subpiles from different piles

541 $\Longleftrightarrow$ $\cancel{4}+1$ $\cancel{4}$ 1 | 330 $\quad$ 2+1 2+1

321 $\Longleftrightarrow$ 2+1 2 1

Call My Bluff card game
(Myerson, Chap. 2)
A chance mechanism selects red or black card
(equally likely) - player 1 observes but <u>not</u>
player 2. Red card is good for player 1.



- One strategy pair is for
player one to raise on red and
fold on black "Rf"
Then optimal response for player 2
is always pass. Mean value is 0,0
Can either player do better?

The <u>normal reprentation</u> of an extensive form game is the strategic form game obtained by having each player select a table of contingincies -- one ~~for~~ for each of the information states at nodes under control of the agent. For the call my bluff card game, this means player one decides which action to take for a red card (i.e. Raise or Fold) and for a black card (i.e. raise or fold). Player 2 Meets or Passes. The payoffs are given by averaging over the leaf nodes:

$$\underbrace{u_i}_{\text{player}}(\underbrace{c}_{\substack{\text{vector of actions for}\\ \text{all information states}}}) = \sum_{\underbrace{x \in \Omega^*}_{\text{leaf notes}}} P(x|c)\, w_i(x)$$

The normal representation of the call my bluff card game:

player one	M	P
Rr	0, 0	1, −1
Rf	.5, −.5	0, 0
Fr	−.5, .5	1, −1
Ff	0, 0	0, 0

eg $A_1(Rf, M) = \frac{1}{2} \cdot 2 + \frac{1}{2}(-1) = 0.5$

$A_1(Rf, P) = \frac{1}{2} \cdot 1 + \frac{1}{2}(-1) = 0$

Let's analyze this static game.

Any strongly dominated strategies? No

weakly " " yes Fr, Ff are weakly dominated by RR

pure strategy NE? no

Suggests $(a, 1-a, 0, 0)$ vs $(b, 1-b)$ $\quad 0<a<1$, $0<b<1$

$(.5)(1-a) = a \qquad (1-b) = (0.5)b$

$a = 1/3 \qquad b = 2/3$

NE (saddlepoint): $\left(\left(\frac{1}{3}, \frac{2}{3}, 0, 0\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$

– Value (player 1) $= \frac{2}{9} \cdot 0 + \frac{1}{9} \cdot 1 + \frac{4}{9} \cdot .5 + \frac{2}{9} \cdot 0 = \frac{1}{3}$

Player 1 always raises on red, raises on black w/p 1/3.

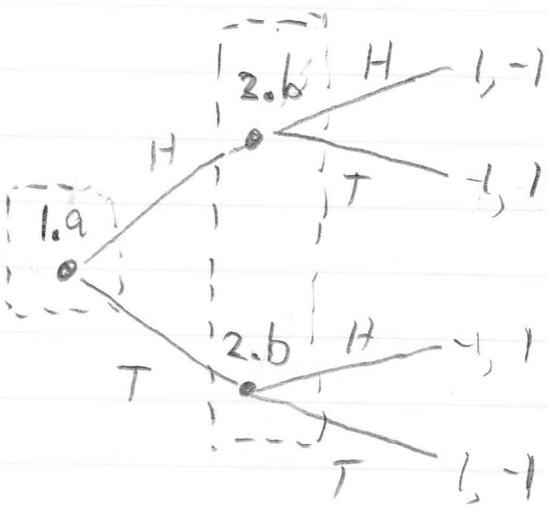
Player 2 meets with prob. 2/3.

Lecture 15
3/12/13

A game in strategic form can be represented as a game in extensive form:

Example: matching pennies

		player 2 H	player 2 T
Player 1	H	1 -1	-1 1
	T	-1 1	1 -1

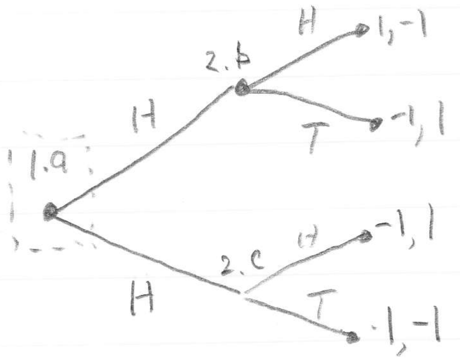


Information state a $\Leftrightarrow$ player 1 knows game is at the beginning

Information state b $\Leftrightarrow$ player 1 has selected move but value is unknown.

Dashed circle indicates grouped nodes have same information state.

If we change the information structure of the previous game:

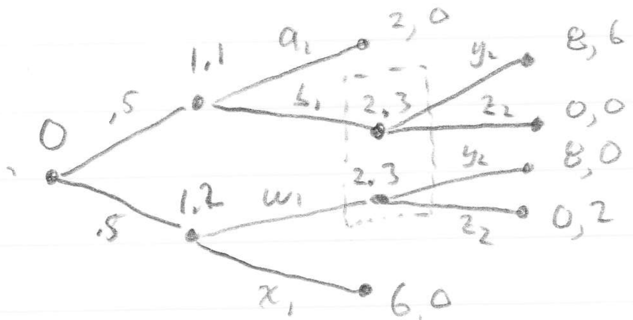


then the normal form of the game becomes

		Player 2			
		HH	HT	TH	TT
Player 1	H	1,-1	1,-1	-1,1	-1,1
	T	-1,1	1,-1	-1,1	1,-1

↑ weakly dominant for player 2 - always wins

Extensive form vs: normal representation :: behavioural representation (Myerson Fig. 2.6)



$$0.5\begin{bmatrix}2,0 & 2,0\\ 2,0 & 2,0\\ 8,6 & 0,0\\ 8,6 & 0,0\end{bmatrix} + 0.5\begin{bmatrix}8,0 & 0,2\\ 6,0 & 6,0\\ 8,0 & 0,2\\ 6,0 & 6,0\end{bmatrix}$$

normal representation (a strategic form game)

Player 1		y_2	z_2	
	$a_1 w_1$	~~5,0~~	~~1,1~~	← strongly dominated
	$a_1 x_1$	4,0	4,0	
	$b_1 w_1$	8,3	0,1	
	$b_1 x_1$	7,3	3,0	

$a_1 w_1$ is strongly dominated by $b_1 x_1$
then y_2 weakly dominates z_2 $\therefore$ (b_1, w_1, y_2) is an NE

multiagent (behavioral) representation

	Player 2	y_2		z_2	
	Agent 1.2 →	w_1	x_1	w_1	x_1
Agent 1.1	a_1	5,5,0	4,4,0	1,1,1	4,4,0
	b_1	8,8,3	7,7,3	0,0,1	3,3,0

Replace player 1 by agents 1.1 and 1.2, each getting same payoff as player 1.

(b_1, y_2, w_1) is a NE

<u>Definition</u> A <u>behavioral strategy profile</u> is any randomized-strategy profile for the multiagent representation of Γ^c.

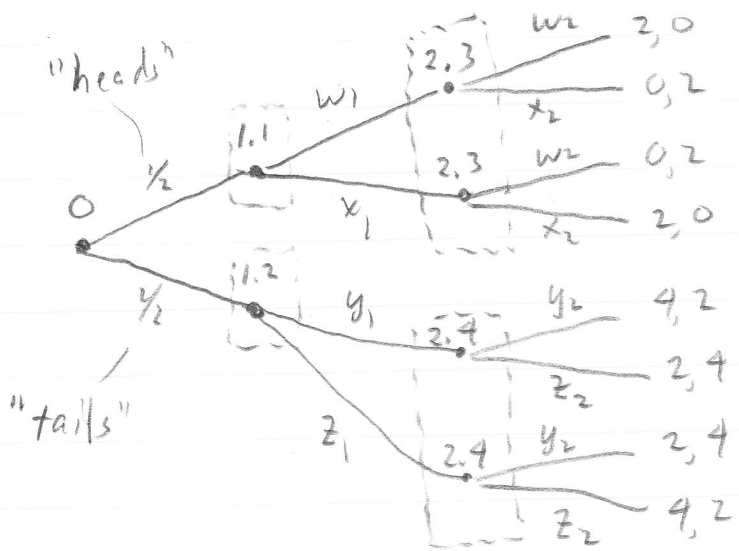
<u>Mixed strategy profiles</u>	<u>Behavioural-strategy profiles</u>
$\underset{i \in N}{\times} \Delta(C_i)$	$\underset{i \in N}{\times} \; \underset{s \in S_i}{\times} \Delta(D_s)$

players

information states for player i

decision set for information state s.

Example (Myerson Fig. 4.1, p. 157)



Normal representation

Player 2

Player 1		$w_2 y_2$	$w_2 z_2$	$x_2 y_2$	$x_2 z_2$
	$w_1 y_1$	3,1	2,2	2,2	1,3
	$w_1 z_1$	2,2	3,1	1,3	2,2
	$x_1 y_1$	2,2	1,3	3,1	2,2
	$x_1 z_1$	1,3	2,2	2,2	3,1

Game is equivalent to first selecting one of the two games using a coin flip:

	w_2	x_2
w_1	2,0	0,2
x_1	0,2	2,0

heads

	y_2	z_2
y_1	4,2	2,4
z_1	2,4	4,2

tails ← same as first game with all payoffs greater by 2.

Both are variations of matching penny's game.

The equilibrium behavioural-strategy profile is:

$$\Big(\underbrace{.5[w_1]+.5[x_1],\ .5[y_1]+.5[z_1]}_{\text{agents for player 1}},\ \underbrace{.5[w_2]+.5[x_2],\ .5[y_2]+.5[z_2]}_{\text{agents for player 2}}\Big)$$

(Much simpler)

– unique

The equilibria in mixed strategy profile form are, for $0 \le \alpha \le 1/2$ and $0 \le \beta \le 1/2$

$$\Big(\alpha[w_1y_1]+\alpha[x_1z_1]+(.5-\alpha)[w_1z_1]+(.5-\alpha)[x_1y_1],$$

$$\beta[w_2y_2]+\beta[x_2z_2]+(.5-\beta)[w_2z_2]+(.5-\beta)[x_2y_2]\Big)$$

α gives correlation between how player 1 would play in two subgames.

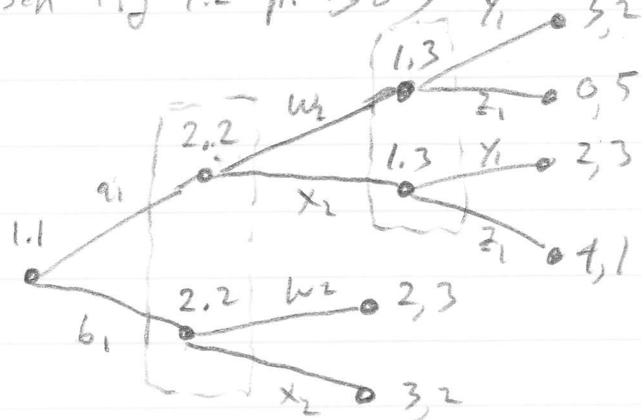
... largely irrelevant, eg. unobservable from multiple plays

β similarly

Lecture 16
3/14/13

Example of computing
a behavioral representation

(Myerson Fig 4.2 p. 158)



$\tau_1 = (.5[a_1 y_1] + .5[b_1 z_1])$ ← A mixed strategy for player 1

What's behavioral representation of τ_1?

$$\sigma_1 = (\sigma_{1,1}, \sigma_{1,3}) = ((0.5, 0.5), (1, 0))$$
$$= (.5[a_1] + .5[b_1], [y_1])$$

Do we simply want equilibrium for multiagent representation?
How about $([b_1], [w_2], [z_1])$? – payoff $(2, 3)$.

Player 1 would like $[a_1]$-$[y_1]$ better.

Normal form of this game.

		.5	.5
		w_2	x_2
.5	$a_1 y_1$	3,2	2,3
	$a_1 z_1$	0,5	4,1
.5	$b_1 y_1$	3,3	3,2
	$b_1 z_1$	2,3	3,2

If player 1 restricts to a_1y_1, b_1y_1 looks like matching pennies

$(.5[a_1y_1] + .5[b_1y_1], .5[w_2] + .5[x_2])$

$((.5, 0, .5, 0), (.5, .5))$ is a NE for the normal form

$\Rightarrow$ $(.5[a_1] + .5[b_1], .5[w_2] + .5[x_2], [y_1])$

is the behavioural representation of the mixed NE

(Myerson p. 161, F+T is similar but not explicit.)

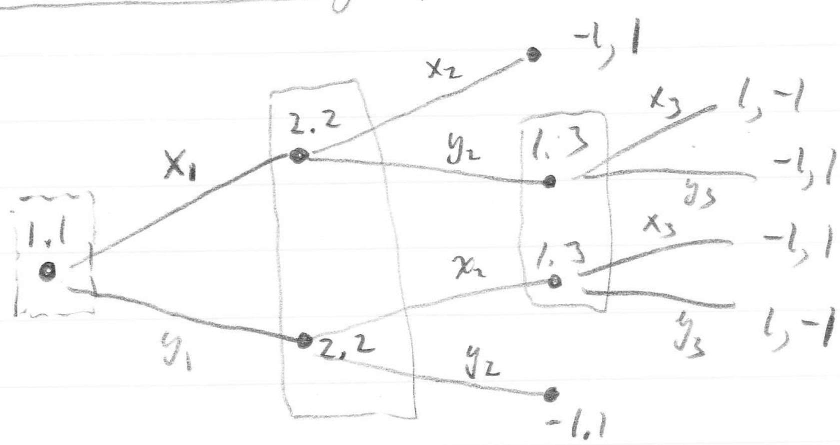
<u>Definition</u> A (Nash) equilibrium of an extensive form game (with perfect recall) (aka equilibrium in behavioral strategies) is a behavioral strategy profile σ such that (1) σ is a NE of the multiagent representation of the game and (2) the mixed representation of σ is a NE of the normal representation.

Perfect recall condition

An extensive form game satisfies the perfect recall condition if whenever a player moves, he remembers all information he knew at earlier positions.

More formally, perfect recall condition is that if nodes x, y, z exist as in this picture, a node w must also exist as in picture, with outgoing branch of same label as shown.

Example violating perfect recall: (Fig 9.3, p. 160 Myerson)



(violates Kuhn's theorem too)

Behavioral reprepresentation (Myerson p.159, Fudenberg + Tirole p. 87)

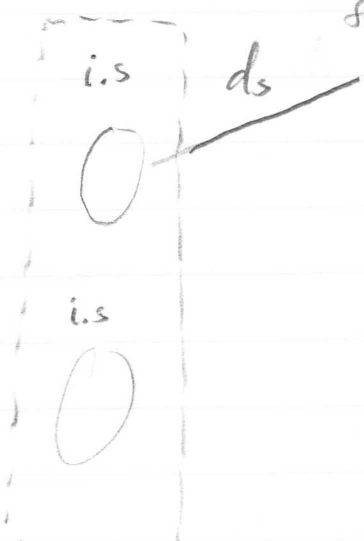
Consider an extensive form game and a player i. A mixed strategy for player i of the normal form game given by $\tau_i \in \Delta(C_i)$. ?

Such a τ_i has a behavioral representation $\sigma_i \in \bigtimes_{s \in C_i} \Delta(D_s)$, required to satisfy:

$$\sigma_{i,s}(d_s)\Big(\sum_{e_i \in C_i^*(s)} \tau_i(e_i)\Big) = \sum_{c_i \in C_i^{**}(d_s, s)} \tau_i(c_i)$$

$C_i^*(s)$: s reachable for some c_{-i}

$C_i^{**}(d_s, s)$: s reachable and branch d_s is taken for some c_{-i}



- Really only makes sense if the perfect recall condition is statisfied.
- Surprising this is enough: behavioural representation is defined not depending on play of other players.

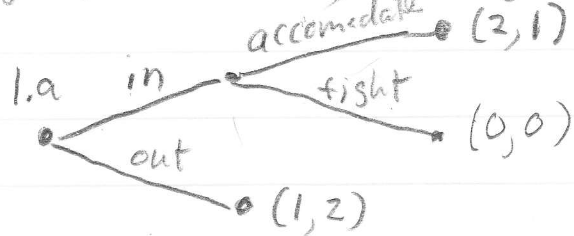
<u>Kuhn's Theorem</u> (Kuhn 1953) If Γ^e is an extensive form game with perfect recall, for any player i, if σ_i is a behavioral representation of a mixed strategy τ_i, for any profile of strategies for the other players τ_{-i}, all players have the same (expected) payoffs under (σ_i, τ_{-i}) and (τ_i, τ_{-i}).

<u>Corollary</u> If Γ^e is an extensive form game with perfect recall, if σ is a behavioral strategy profile and τ a mixed strategy profile of the normal representation of Γ^e, and σ_i is a behavioral representation for τ_i for all i, then σ is an NE of Γ^e iff τ is an NE of the normal representation of Γ^e.

<u>Corollary</u> NE exists for extensive game with perfect recall.

Entry Deterence Game

Player 1 = entrant, Player 2 = encumbant



normal representation

a NE

		player 2	
		accom.	fight
player 1	in	2,1	0,0
	out	1,2	1,2

this is a NE too
(what does it mean?

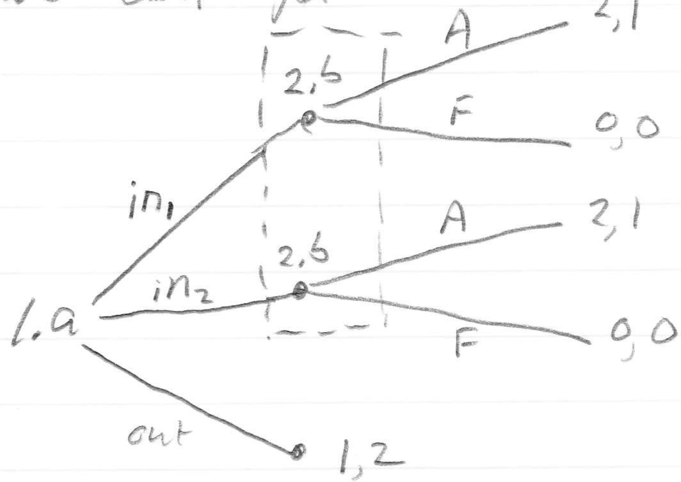
Of the two NE,
only (2,1) is subgame perfect.

A (proper) subgame is defined by an information state S consisting of a single node x, so that for any node y following x, all nodes with the same information state as y also follow x. A strategy profile σ is <u>subgame perfect</u> if σ_i is a best response to σ_{-i} for any (proper) subgame.

If we slightly modify entry deterrence game,

we can get:

A = accomodate
F = fight



Note that actions in$_1$ and in$_2$ are equivalent. But now there is no subgame for player 2 begining at a node. So (out, F) is subgame perfect.

To rule it out, Krebs + Wilson ('82) defined sequential equilibrium in which games starting at information states with multiple nodes are considered. (as well as subgames from single node information states.)

<u>Sequential equilibria</u> (strengthening of subgame perfect equilibria)
Recall that in an extensive form game, an information state s is a set of nodes grouped by dashed lines. It is a maximal set of nodes with the same information state. All the nodes in an information state are controlled by a single player. A <u>behavioral strategy profile</u> $\sigma = (\sigma_i)_{i \in I}$ consists of a probability distribution $(\sigma_i(d|s) : d \in D_s)$ over the decision set D_s for every player i and every information state s for i.
A <u>belief vector</u> μ consists of a probability distribution $(\mu(x|s) : x \in s)$ of s for every information state s.

An pair (σ, μ) consisting of a behavioral strategy profile σ and a belief vector μ is called an <u>assessment</u>.

For an extensive form game with imperfect information, payoffs are determined by the terminal node that is reached. It is therefore easy to define

$u_i(\sigma \mid x)$ (payoff for player i given node x is reached.)

such payoffs only depend on decisions made at x and later in the game. For an information state s and belief vector $\mu(s)$ we define

$$u_i(\sigma \mid s, \mu(s)) = \sum_{x \in s} u_i(\sigma \mid x)\, \mu(x \mid s)$$

<u>Definitions</u>

(S) An assesment (σ, μ) is <u>sequentially rational</u> if for any player i, any information state s of player i, and any alternative behavioural strategy σ_i' for player i,

$$u_i(\sigma \mid s, \mu(s)) \geq u_i((\sigma_i', \sigma_{-i}) \mid s, \mu(s))$$

Let Σ_i^0 be the set of fully mixed behavioral strategies and let $\Psi^0 = \left\{ (\sigma, \mu) : \sigma \in \Sigma_i^0, \ \mu_{i,s}(x) = \frac{P(x \mid \sigma)}{\sum_{x' \in s} P(x' \mid \sigma)} \right\}$

(C) An assessment (σ, μ) is <u>consistent</u> if there exists $(\sigma^{(n)}, \mu^{(n)}) \in \Psi^0$ with

$$(\sigma, \mu) = \lim_{n \to \infty} (\sigma^{(n)}, \mu^{(n)})$$

<u>Def</u> A <u>sequential equilibrium</u> is an assessment (σ, μ) satisfying S and C.

<u>Def</u> σ is a <u>sequential-equilibrium scenario</u> if (σ, μ) is a sequential equilibrium for some μ.

(Setting: Extensive form game with perfect recall.)

Proposition If σ is a (trembling hand) perfect equilibrium of the multi agent representation of Γ^e, then σ is a sequential-equilibrium-scenario!

(Myerson p. 217)

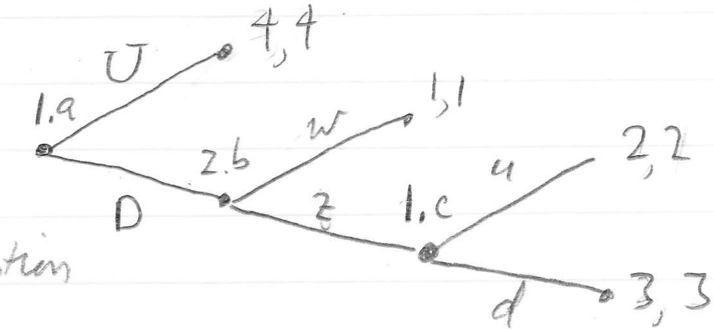
Proof By definition there is a sequence $(\sigma^{(n)})$ of fully mixed profiles with $\sigma^{(n)} \to \sigma$ and $\sigma_i \in B(\sigma_{-i}^{(n)})$ for all n. Let $\mu_{i,s}^{(n)}(x) = \dfrac{P(x|\sigma^{(n)})}{\sum_{x' \in S} P(x'|\sigma^{(n)})}$

(i.e. $\mu^{(n)}$ is the unique belief vector determined by $\sigma^{(n)}$ and Bayes rule for each information state s.

By compactness, going to a subsequence if necessary, we can assume $\lim_{n \to \infty} \mu^{(n)}$ exists. Let limit be μ.

Then (σ, μ) is a sequential equilibrium ((s) follows using continuity of u_i and letting $n \to \infty$ in $u_i((\sigma_i, \sigma_{-i}^{(n)})|s, \mu^{(n)}(s)) \geq u_i((\sigma_i', \sigma_{-i}^{(n)})|s, \mu^{(n)}(s))$. for any behavioral profile σ'

Example
Illustrates condition in previous proposition



Extensive form

	w	z
Uu	4,4	4,4
Ud	4,4	4,4
Du	1,1	2,2
Dd	1,1	3,3

(Uu, z) is a trembling hand perfect equilibrium of the normal representation of game. While σ defined by $\sigma = ([U], [z], [u])$ is a behavioral representation of τ, it is only an NE, not a THP equilibrium, of the multiagent version of the game. Problem is, under Uu, player 1 doesn't recognize possibility he trembled at state 1.a when considering 1.c.
Note: $([U], [z], [d])$ is THP for multiagent game.

<u>Note</u> If an extensive form game has the property that no player can play twice (or more) in any instance of the game, then if τ and τ' are mixed strategies for the normal form game both equivalent to the same behavioural strategy σ, then there is no way to distinguish among τ, τ', or σ when watching the game. Moreover, τ is THP equilibrium for the extensive form representation of the game <u>iff</u> σ is a THP equilibrium for the multiagent representation. $\therefore$ If τ is THP, σ is a sequential-equilibrium-scenario.

Proposition (Selton '75) A finite strategic form game $(I, (A_i : i \in I), (u_i)_{i \in I})$ has at least one trembling-hand perfect equilibrium.

Corollary Every finite extensive form game has at least one sequential-equilibrium-scenario σ.

Notation For a finite set F and $\varepsilon > 0$ with $\varepsilon |F| \leq 1$, let $\Delta_\varepsilon(F)$ denote the set of probability vectors p over F with $p(a) \geq \varepsilon$ for all $a \in F$.

Proof of proposition For $\varepsilon > 0$ small enough, consider the ε-perturbed game with payoff functions $u_i(\tau_i, \tau_{-i})$ for $\tau_i \in \Delta_\varepsilon(A_i)$. Let $\varepsilon_n \to 0$. For each n sufficiently large there is a NE profile $\tau^{(n)}$ for the ε_n-perturbed game (by fixed point theorem as used to prove Nash's theorem).

By compactness, going to a subsequence of n if necessary, it can be assumed $\lim_{n\to\infty} \tau^{(n)}$ exists. Let $\tau = \lim_{n\to\infty} \tau^{(n)}$. Note that $\tau = (\tau_i)_{i\in I}$ is a mixed strategy profile. Let $A_i^* = \{a_i \in A_i : \tau_i(a_i) > 0\}$, i.e. A_i^* is the support set for τ_i. By going to a subsubsequence if necessary, it can be assumed $\tau_i^{(n)}(a_i) > \varepsilon^{(n)}$ for all $i \in I$, all $a_i \in A_i^*$, and all n. Thus, $A_i^* \subset B_i(\tau_{-i}^{(n)})$ for all $i \in I$ and all n. Therefore, τ_i is a best response to $\tau_{-i}^{(n)}$ for all i, n, and $\tau^{(n)}$ is a profile of fully mixed strategies with $\tau^{(n)} \to \tau$. So τ is trembling hand perfect. $\square$

Problem: Find a polynomial time algorithm to compute a trembling hand perfect equilibrium for a two person zero-sum game.

<u>Proposition</u> In a <u>two</u> <u>player</u> matrix game, a strategy profile $\tau = (\tau_1, \tau_2)$ in mixed strategies is THP if and only if neither τ_1 nor τ_2 is weakly dominated (here p weakly dominates p' for a player i if $u_i(p, j) \geq u_i(p', j)$ with strict inequality for at least one value of j.) (See, eg Osbourne + Rubenstein)