Potential Games (1996)
(Monderer + Shapley)

Let $G = (I, (S_i), (u_i))$ be a strategic form game
$S = \times_{i \in I} S_i$

Definition A potential function for G

is a function $\Phi : S \to \mathbb{R}$ such that

$$u_i(x, s_{-i}) - u_i(y, s_{-i}) = \Phi(x, s_{-i}) - \Phi(y, s_{-i})$$
for all $i \in I$, $x, y \in S_i$

- Easy to find *if* one exists.
- If Φ is a potential function, so is $\Phi + c$

Example

Prisoners dilemma (lecture 2)

	L	R
L	1, 1	−1, 2
R	2, −1	0, 0

$\xrightarrow{\text{derive}}$

Φ	L	R
L	0	1
R	1	2

Is there a potential function?
set $\Phi(L, L) = 0$

Φ :	L	R
L	0	
R		

must be 1

must be 2 again

must be 1
why?

so must be 2
why?

Φ	L	R
L	0	1
R	1	2

$\therefore$ works.

Suppose $s = (s_i)_{i \in I}$ is a strategy vector, and suppose for some player i_0 s_i is replaced by s_i' such that

$$u_i(s') > u_i(s) \quad \text{where } s' = (s_i', s_{-i})$$

(i.e. the response of a single player is changed and is better for that player)

Then $\Phi(s') > \Phi(s)$.

$\therefore$ <u>Proposition</u> Let G have potential Φ

(i) (s is a pure strategy NE) $\Longleftrightarrow$ s is a local max of Φ (includes global) (where neighbors of s are vectors obtained by single player changes)

(ii) If G is finite then $\arg\max_s \Phi(s) \neq \emptyset$ and pure strategy NE's exist.

(iii) If G is finite, a sequence $s^0, s^1, \ldots$ of strategy vectors obtained by single player better response dynamics terminates in a finite number of steps, ends at a NE.

so what? $\rightarrow$

(single resource congestion game)

<u>Example</u> Let $S_i = \{0,1\}$ and $u_i(s) = s_i\, c(\|s\|_1)$

for some function $c: \{0, \dots, |I|\} \to \mathbb{R}$.

Is this a potential game?

Need for any i, s_{-i}

$$u_i(1, s_{-i}) - u_i(0, s_{-i}) = \Phi(1, s_{-i}) - \Phi(0, s_{-i})$$

Suppose s_{-i} has k ones. This becomes

$$c(k+1) - 0 = \underbrace{\Phi(1, s_{-i})} - \underbrace{\Phi(0, s_{-i})}$$

Represents an arbitrary pair of strategy vectors s', s $|s| = k$, $|s'| = k+1$, $s' \geq s$.

can set to zero

$$\therefore\ \Phi(s) = \Phi(0) + c(1) + \cdots + c(\|s\|_1),\quad s \neq (0, \cdots, 0)$$

Often $c(1) > c(2) \cdots > c(k) \geq 0$, represents congestion cost. Not necessary though. Could also add separable linear terms:

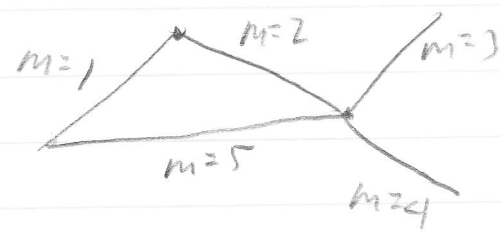
$$u_i(s) = s_i\{c(\|s\|_1 - p_i\}\qquad \Phi(s) = c(1) + \cdots + c(\|s\|_1) - \sum_i s_i p_i$$

Example

More generally we have a congestion game:

$\mathcal{I} = \{1, \dots I\}$ = set of players

$\mathcal{M} = \{1, \dots, m\}$ = set of resources (eg. links)



Routing problem

$$S_i \subset 2^{\mathcal{M}} \quad \text{eg. } S_1 = \{\emptyset, \{1,2,3\}, \{5,3\}\}$$

$$S_2 = \{\emptyset, \{2,4\}\}$$

$$\vdots$$

$$u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j) \qquad k_j = \text{\# of players using resource } j$$

$$\text{Potential: } \Phi(s) = \sum_j \sum_{k: 1 \le k \le k_j} c^j(k)$$

Proof that Φ is a potential — both Φ and u_i's are sums over resources. For each resource, reduces to previous example.

Generalization $u_i(s_i, s_{-i}) = \varphi_i(s_i) + \sum_{j \in s_i} c^j(k_j)$

(Add $\sum_j \varphi_j(s_j)$ to Φ)

While it is easy to see if a given game has a potential function, unfortunately not too many do. A weaker property with similar consequences:

Def Φ is an ordinal potential function for $(I, (S_i), (u_i))$ if

$$u_i(x, s_{-i}) > u_i(y, s_{-i}) \Rightarrow \Phi(x, s_{-i}) > \Phi(y, s_{-i})$$

for all i, $x, y \in S_i$

Propositions about finite games extend to ordinal potential functions.

Example Cournot competition: $q_i \in (0, +\infty)$

$$u(q_i, q_{-i}) = q_i \left(P(Q) - c\right) \quad Q = q_1 + \cdots + q_I$$

$$\Phi(q) = \left(\prod_j q_j\right)\left(P(Q) - c\right)$$ ← easily seen to be an ordinal potential function.

- Could make a finite game if S_i replaced by finite subset of $(0, +\infty)$ for each i.

Prediction with expert advice (C.-B. + L. 2.1)

Suppose a forecaster selects an action $\hat{p}_t \in \mathcal{D}$ for each $t \geq 1$ in connection with a sequence of outcomes $(y_t : t \geq 1)$ with $y_t \in \mathcal{Y}$.

For a loss function $\ell : \mathcal{D} \times \mathcal{Y} \to \mathbb{R}$

$$\hat{L}_n = \underset{\text{forecaster}}{\overset{\text{cumulative}}{\text{loss of}}} = \sum_{t=1}^{n} \ell(\hat{p}_t, y_t)$$

For example, the problem may be to predict y_t.

Here the sequence $y_1, y_2, \cdots$ can be arbitrary.

Could have $y_t = g_t(\hat{p}_1, \cdots, \hat{p}_{t-1}, y_1, \cdots, y_{t-1})$.

g_t could be totally unknown
- if known to be in a class, could maybe try to learn it.

Idea Suppose there are N experts.

Expert i makes prediction $f_{i,t}$ at time t. Suffers cumulative loss:

$$L_{i,n} \triangleq \sum_{t=1}^{n} \ell(f_{i,t}, y_t)$$

regret $R_{i,t} \triangleq \hat{L}_n - L_{i,n} = \sum_{t=1}^{n} \underbrace{\ell(\hat{p}, y_t) - \ell(f_{i,t}, y_t)}_{}$

(Regret of forecaster for not following expert i.) $r_{i,t}$ = instantaneous regret.

Negative is better!

Reasonable (?) goal:

$$\max_{1 \le i \le N} \frac{1}{n}(\hat{L}_n - L_{i,n}) \xrightarrow{n \to \infty} 0$$

A universal type result. Forecaster seeks average loss of best forecaster.

We'll see it can be done if $\ell(p, y)$ is a bounded <u>convex</u> function of p.

<u>Important lemma</u> (Assume $\ell(p,y)$ convex in p)

If $\hat{p}_t = \dfrac{\sum_i w_{i,t-1} f_{i,t}}{\sum_{i'} w_{i',t-1}}$ for some weights $(w_{i,t-1})$

$\sum_{i'} w_{i',t-1} = D_{t-1}$

then $\underbrace{\sum_i w_{i,t-1} r_{i,t}}$ ≤ 0 for any y_t.

instantaneous regrets

<u>Proof</u> For any y_t

$$\ell(\hat{p}_t, y_t) = \ell\left(\sum \left(\frac{w_{i,t-1}}{D_{t-1}}\right) f_{i,t}, y_t\right) \quad \text{Jensen's inequality}$$

$$\le \sum_i \frac{w_{i,t-1}}{D_{t-1}} \ell(f_{i,t}, y)$$

so

$$\sum_i \frac{w_{i,t-1}}{\cancel{D_{t-1}}} \{\ell(\hat{p}_t, y_t) - \ell(f_{i,t}, y) \le 0 \qquad \boxtimes$$

Interpretation. Forecaster can do at least as well as any given average of the experts. (Intuitively would like to give higher weight to better forecasters.)

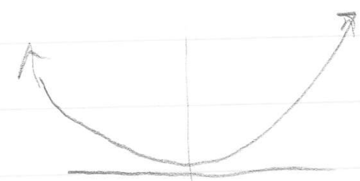
Can't make all regrets negative. But can insure a specified weighted average ≤ 0.

Lecture 11
2/19/13

Let $\Phi(x_1, \cdots, x_N) \equiv \frac{1}{\eta}\ln\left(\sum_{i=1}^{N} e^{\eta x_i}\right)$

- $\Phi(x) \geq x_i$

<u>Note</u> • $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x_1} \\ \vdots \\ \frac{\partial \Phi}{\partial x_n} \end{pmatrix} = \begin{pmatrix} e^{\eta x_1} / \sum e^{\eta x_i} \\ \vdots \\ e^{\eta x_n} / \sum e^{\eta x_i} \end{pmatrix} = \sigma(\eta x_i) \sim$ σ is soft max function

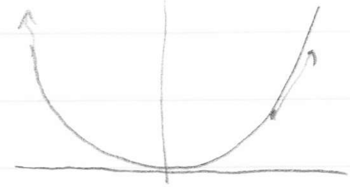


- $\lim_{x_i \to +\infty} \Phi(x_i, x_{-i}) / x_i = 1$

- $H(\Phi)\big|_x = \left(\frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right) = \eta\left(\mathrm{diag}\left(\frac{e^{x_i}}{D} \right) - \left(\frac{e^{x_i}}{D} \frac{e^{x_j}}{D} \right) \right) > 0$

(positive definite because diagonal elements are strictly positive and diagonal is strictly greater than sum of off diagonal.

We'll use $\Phi(\underbrace{R_{1,t}, \ldots, R_{N,t}}_{\underline{R}_t = \underline{R}_{t-1} + r_t})$ as a Lyapunov function. $\quad r_{i,t} = \ell(\hat{p}, y_t) - \ell(i, y_t)$



$$\Phi(R_t) = \Phi(R_{t-1} + r_t)$$

$$= \Phi(R_{t-1}) + \underbrace{\nabla\Phi(R_{t-1})}\cdot r_t + \tfrac{1}{2} r_t^T H(\Phi)\big|_\xi r^T$$

$\hat{p}_t = \left(\frac{e^{\eta R_{1,t-1}}}{D_{t-1}}, \ldots, \frac{e^{\eta R_{N,t-1}}}{D_{t-1}}\right)$: use as weight vector. Then this term is negative. "Blackwell's condition"

some $\xi \in \mathbb{R}^n$ on line from R_{t-1} to $R_{t-1} + r_t$

$\boxed{\text{Assume } \ell(p, y) \in [0,1]}$

Suppose $\ell(p, y) \in [0,1]$ for all p, y.

Then $r_t^T H(\Phi)\big|_\xi r_t \le \eta \sum_i \left(\frac{e^{\eta \xi_i}}{\sum_{i'} e^{\eta \xi_{i'}}}\right) r_{i,t}^2 \le \eta$.

so $\Phi(R_n) \le \Phi(R_0) + \frac{n\eta}{2}$ $\qquad \Phi(\overset{0}{R_0}) = \frac{1}{\eta}\ln(\Sigma e^0) = \frac{\ln(N)}{\eta}$

$$\Phi(R_n) \le \frac{\ln(N)}{\eta} + \frac{n\eta}{2}$$

so $\max_i R_{n,i} \le \frac{\ln(N)}{\eta} + \frac{n\eta}{2} = \sqrt{2n\ln N}$ ← # of experts

(can tighten to 8)

(for any $y_1, \ldots y_n$) $\quad \eta = \sqrt{2\ln N/n}$

<u>Connection to discrete-time purturbed fictitious play.</u>

If we let $\eta = \frac{\gamma}{t}$ (time varying) this is the discrete-time version of fictitious play we considered earlier, but using the soft max σ.

Indeed, we see that $\frac{1}{n}\sum_{t=1}^{n} \ell(i, y_t)$

is the payoff for pure strategy i against the empirical average of (joint) strategies

(Here we assume the y_t's are

can remove $\hat{L}$ terms if D

and

$$\hat{P}_{t+1} = \left(\frac{e^{\eta(\hat{L}_t - L_{1,t})}}{D_{t+1}}, \dots, \frac{e^{\eta(\hat{L}_t - L_{N,t})}}{D_{t+1}} \right)$$

$$= \left(\frac{e^{-\eta L_{1,t}}}{\tilde{D}_{t+1}}, \dots, \frac{e^{-\eta L_{N,t}}}{\tilde{D}_{t+1}} \right) \qquad \tilde{D}_t = D_t e^{-\eta \hat{L}_{t-1}}$$

$$= \sigma\left(-\gamma \tfrac{1}{t} L_t\right) \qquad \sigma(x) = \left(\frac{e^{x_i}}{\sum_{i'} e^{x_{i'}}} \right)_{1 \le i \le N}$$

Application to finite player games. (C-B. + L Sect. 4.2)

Consider a player of a game with a finite space of N pure strategies, indexed by $1 \le i \le N$.

and loss function $\ell(i, y_t)$
↖ plays of other players.

Let D = space of mixed strategies
(probability vectors $(p_1, \cdots, p_N)$.

Then $\ell(p, y_t) = \sum_i p_i \ell(i, y_t)$. denotes expected loss for play p against y_t.

- D is convex
- $\ell(p, y_t)$ is convex (actually, linear) in p.

Let "experts" be the N pure strategies.

Corollary: If player uses the exponentially weighted strategies: $\hat{p}_{t+1} = \left(\dfrac{e^{-\eta \sum_{s=1}^{t} \ell(i, y_s)}}{D_t} \right)_{1 \le i \le N}$ where $\eta > 0$
and $\ell(\hat{p}_t, y_t) \in [0,1]$
then

$$\underbrace{\sum_{t=1}^{n} \ell(\hat{p}_t, y_t)}_{\bar{L}_n \text{ = sum of expected losses}} - \min_i \sum_{t=1}^{n} \ell(i, y_t) \le \frac{\ln N}{\eta} + \frac{n\eta}{8}$$

(for any $y_1, \cdots y_n$)

- y's could be random. eg. driven by $I_1, I_2, \cdots$

2-page time-out -- we'll need the following

Azuma-Hoeffding inequality (ECE 534 notes Sect. 10.3)

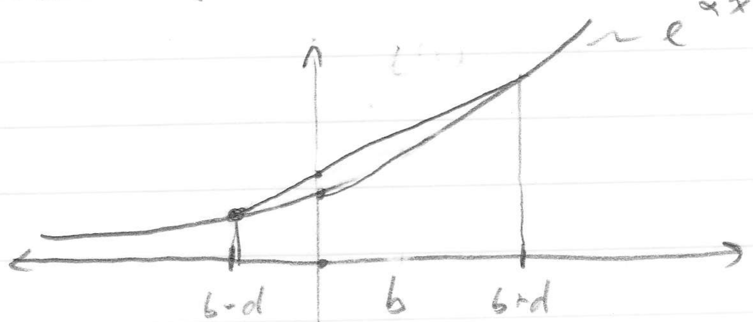
Lemma Suppose D is a random variable with

$$E[D]=0 \quad \text{and} \quad D \in \underbrace{[b-d,\, b+d]}_{\text{interval of length } 2d = c}$$

Then

$$E[e^{\alpha D}] \le e^{(\alpha d)^2/2}$$

Proof



$$E[e^{\alpha D}] \le E[L(D)] = \frac{d-b}{2d} e^{\alpha(b+d)} + \frac{d-b}{2d} e^{\alpha(b-d)}$$

$$\text{so } \underbrace{\ln(\ \text{"}\ )}_{\frac{d^2}{d\alpha^2}(\ ) \le 1} \le \frac{\alpha^2 d^2}{2}$$

$\square$

Lecture 12 R9
2/27/13

Azuma-Hoeffding Inequality (with centering)

Let $(D_n : n \geq 1)$ be a martingale difference sequence: (i.e. $E[D_n \mid D_1, \cdots, D_{n-1}] = 0$)

such that for some predictable sequence B_n (so $B_n = f(D_1, \cdots, D_{n-1})$)

$|D_n - B_n| \leq d_n$ for all n. (with probability one)

Then

$$P\{ \sum_{t=1}^{n} D_t \geq \lambda \} \leq \exp\left(-\frac{\lambda^2}{2 \sum d_t^2}\right)$$

Proof

$$E[e^{\alpha(\sum_{t=1}^{n} D_t)}] = E[e^{\alpha D_t} E[e^{\alpha(\sum_{t=1}^{n-1} D_t)} \mid D_1, \cdots, D_{n-1}]]$$

$$\leq e^{\frac{\alpha^2 d_n^2}{2}} E[e^{\alpha(\sum_{t=1}^{n-1} D_t)}] \cdots \leq e^{\frac{\alpha^2}{2} \sum_{t=1}^{n} d_t^2}$$

Repeat.

Markov inequality:

$$P\{ \sum_{t=1}^{n} D_t \geq \lambda \} \leq E[e^{\alpha\{\sum_{t=1}^{n} D_t - \lambda\}}] \leq e^{\frac{\alpha^2}{2} \sum d_t^2 - \alpha\lambda}$$

Take $\alpha = \frac{\lambda}{\sum d_i^2}$

Corollary extends to random Y's

with $Y_t = g(I_1, \cdots, I_{t-1})$, where I_s is the actual play of the player at time s. Y_t can also depend on external randomness, such as random choice of J_t's for other player. Then

$$\sum_{t=1}^{n} \ell(\hat{p}_t, Y_t) - \min_i \sum_{t=1}^{n} \ell(i, Y_t) \le \frac{\ln N}{\eta} + \frac{n\eta}{8}.$$

a probability one result.

We can also address actual (random) payoff of the player

$$\sum_{t=1}^{n} \ell(I_t, Y_t) - \min \sum_{t=1}^{n} \ell(i, y_t)$$

because

$$\left(\ell(I_t, Y_t) - \ell(\hat{p}_t, Y_t)\right)_{t=1}^{n} \text{ is a}$$

martingale difference sequence (relative to $\sigma(I_1, \cdots, I_t)$) with conditional range length one

$$[0 - \ell(\hat{p}_t, Y_t), 1 - \ell(\hat{p}_t, Y_t)]$$

So Azuma-Hoeffding bound yields

$$P\left\{ \sum_{t=1}^{n} \ell(I_t, Y_t) - \ell(\hat{p}_t, Y_t) \geq \gamma \right\} \leq e^{-\frac{2\gamma^2}{n}}$$

$$= \delta$$

$$e^{-\frac{\gamma^2}{2n}} = \delta$$

$$\frac{2\gamma^2}{n} = -\ln \delta$$

$$\gamma = \sqrt{\frac{n}{2} \ln(1/\delta)}$$

$\therefore$ With probability at least $1-\delta$, under the exponential rule,

$$\sum_{t=1}^{n} \ell(I_t, Y_t) - \min_i \sum_{t=1}^{n} \ell(i, Y_t) \leq \frac{\ln N}{\eta} + \frac{n\eta}{8} + \sqrt{\frac{n}{2} \ln\left(\frac{1}{\delta}\right)}$$

Thus, the time averaged regret (divide by n)

converges to zero a.s. as $n \to \infty$.

$\ln(\frac{1}{\delta}) = \ln(n^4)$ (Need to apply Borel-Cantelli lemma.)

$$\left\{ \ln\left(\tfrac{1}{\delta}\right) = n^{1-\varepsilon} \Rightarrow \delta_n = e^{-n^{1-\varepsilon}} \quad \sum_{n=1}^{\infty} P\{\text{bound not met}\} < +\infty \right.$$

so $P\{$inequality violated infinitely often$\} = 0$.

<u>Definition</u> A forecaster is Hannan consistent if

$$\limsup_{n\to\infty} \frac{1}{n}\left(\sum_{t=1}^{n} \ell(I_t, Y_t) - \max_{i=1,\dots,N} \sum_{t=1}^{N} \ell(i, Y_t)\right) = 0 \text{ a.s.}$$

We've nearly* shown the following: The exponentially weighted average forecaster with $\eta_t = \sqrt{(8\ln N)/t}$ is Hannan consistent.

(see C-L for other details)

*part we skipped is analysis for time varying η_t.

Repeated
Two player zero sum game.

C.-B. + L., Sect. 7.3

Consider two player game. ℓ

Player one wants to select i to minimize $\ell(i,j)$

" two " " " j to maximize $\ell(i,j)$

Suppose players play game repeatedly, player one using a Hannan consistent forecaster. (e.g. exponential weighted play with weight $e^{-\eta_t \sum_{s=1}^{t-1} \ell(i, J_s)}$)

Then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) \le V \quad \text{a.s.}$$

value of game.

Proof By def^n of Hannan consistent, suffices to prove $\limsup_{n\to\infty} \min_i \frac{1}{n}\sum_{i=1}^{n} \ell(i, J_t) \le V$ a.s.

or $\limsup_{n\to\infty} \min_p \ell(p, \hat{q}_t) \le V \qquad \hat{q}_{j,t} = \frac{1}{n}\sum_{t=1}^{n} I_{\{J_t = j\}}$

But $\min_p \ell(p, \hat{q}_t) \le \max_q \min_p \ell(p,q) = V$. $\square$

<u>Corollary</u> If both players in repeated play of a zero sum game use a Hannan consistent forecaster,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) = V \quad \text{a.s.}$$

<u>Proof</u> $\limsup (\ ) \le V$ and $\liminf (\ ) \le V$ a.s.

<u>Corollary</u> If both players use Hannan consistent forecasters, (for a zero sum game) any limit point p^* of the marginal empirical distribution of play $\hat{p}_n$, empir..

$$\hat{p}_n(i) \triangleq \frac{1}{n}\left|\{1\le t\le n : I_t = i\}\right| \text{ is minmax optimal.}$$

(If q^* is similarly a limit point for player 2, (p^*, q^*) is a saddle point.)

<u>Proof</u> Let p^* be a limit point of $\hat{p}_n$.

Then $$\max_j \ell(p^*, j) \underset{p^* \text{ close to } \hat{p}_n}{\le} \max_j \ell(\hat{p}_n, j) + \varepsilon = \max_j \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, j) + \varepsilon$$

Hannan consistency for player 2, for n large enough depending on ω.

$$\le \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) + 2\varepsilon \le V + 3\varepsilon$$

by Corollary above, for n large enough.

so p^* is min max optimal.

Similarly q^* is max min optimal for player two.

$\therefore (p^*, q^*)$ is an NE/saddle point.

Notes

- Corollary does not claim that limit points of $\hat{p}_t$ (distribution used by player one at time t) are minimax optimal.

- Similarly, limit points of the joint empirical distⁿ

$$p_{n,\text{empirical}}(i,j) = \frac{1}{n}\left|\{t : 1 \le t \le n, (I_t, J_t) = (i,j)\}\right|$$

are not necessarily product distributions.

Limit points of the joint distⁿ are Hannan consistent:

$$\sum_i \sum_j p(i,j)\,\ell(i,j) \ge \sum_i \sum_j p(i,j)\,\ell(i',j) \quad \forall i'$$

(may not even be a correlated equilibrium; Can use enhanced versions that are "internally consistent") and then for multiple player games (not nec. zero sum) joint empirical distⁿs converge to set of correlated equilibria. i.e those so that

$(i \to i')$ $$\sum_j p(i,j)\,\ell(i,j) \ge \sum_j p(i,j)\,\ell(i',j) \quad \forall i, i'$$

Blackwell Approachability Theorem

We begin by a way to think about the minmax and maxmin performace a a player can expect from a game.

(From ps. 1 #6

	1	2	
1	0,0	1,2	1,3
2	2,1	0,0	2,3
	3,1	3,2	0,0

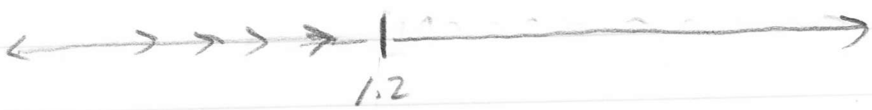
)

Forget about payoffs of player 2. So payoffs of player one are:

1	0	1	1
2	2	0	2
3	3	3	0

The max min strategy for player one is $(0, 0.4, 0.6)$ with payoff 1.2. (Proof: If player 2 directly opposes player 1, $((0,0.4,0.6), (0,0.4,0.6))$ is a saddle point.)

So player one can force upward drift on empirical average loss; in region $(-\infty, 1.2)$.



The minimax strategy is $(1,0,0)$, payoff 1

Proof: If player 2 directly opposes player 1, (i.e. minimizes)

$((1,0,0), (0, 0.4, 0.6))$ is a saddlepoint.

So player one can force downward drift on empirical average loss, in region $(1.0, +\infty)$

<u>Claim</u> For any $x_0 \in [1.0, 1.2]$, player 1 can play so that $\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) = x_0$ for any choice of $(\hat{q}_t : t \geq 1)$ by player 2.

(at time t, player 1 selects $\hat{p}_t$ based on past outcomes.

Also, player 2 can ensure that

$$\lim_{n\to\infty} d\left(\frac{1}{n}\sum_{t=1}^{n} l(I_t, J_t), [1.0, 1.2]\right) = 0$$

distance between empirical average loss and the interval [1.0, 1.2]

<u>Definition</u> A set $A \in \mathbb{R}$ is <u>approachable</u> for a player if the player can play to insure $d\left(\frac{1}{n}\sum_{t=1}^{n} l(I_t, J_t), A\right) \xrightarrow[n\to\infty]{} 0$ a.s.

For above example, if $A \subset \mathbb{R}$, and A is closed:

* A is approachable for player 1 $\iff A \cap [1, 1.2] = \emptyset$
* A is approachable for player 2 $\iff [1, 1.2] \subset A$

In particular,

$[v, +\infty)$ is approachable for player one $\iff v \le 1.2$
$(-\infty, u]$ " " " " $\iff u \ge 1.0$
$[v, +\infty)$ " " " two $\iff v \le 1.0$
$(-\infty, u]$ " " " " $\iff u \ge 1.2$

In some instances, player has multiple objectives, such as keeping regret small vs. multiple experts.

So let $\ell(i,j)$ be vector valued $\ell(i,j) \in \mathbb{R}^m$, $\|\ell(i,j)\|_2 \leq 1$

Def A set $S \in \mathbb{R}^m$ is approachable if player 1 has a (randomized) strategy so that, no matter how the other player plays,

$$\lim_{n \to \infty} d\left(\frac{1}{n} \sum_{t=1}^{n} \ell(I_t, J_t), S \right) = 0$$

$$d(u,S) = \inf_{v \in S} \|u-v\|_2$$

(wlog, only consider closed subsets of unit ball in $\mathbb{R}^m$.)

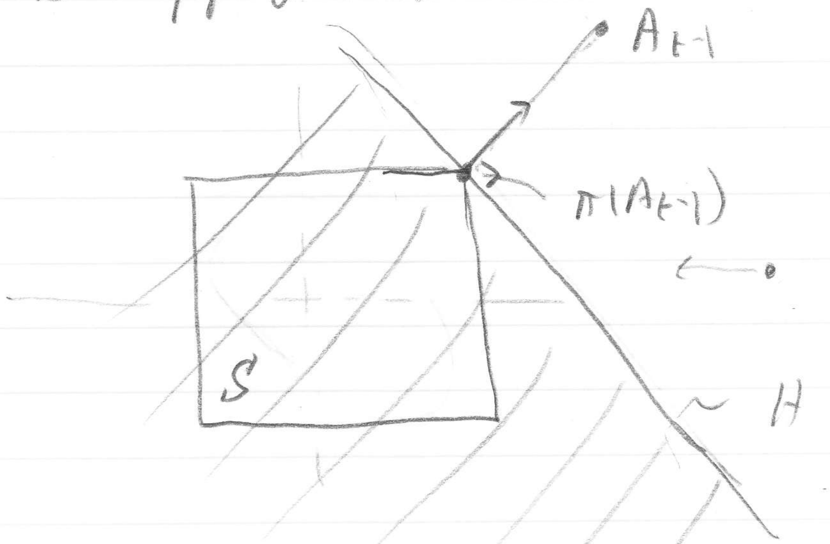
Lemma A half space $\mathcal{H} = \{\underline{u} : \underline{a} \cdot \underline{u} \leq c\}$ is approachable iff for some p,

$$\max_j a \cdot \bar{\ell}(p, j) \leq c$$

(Simple -- reduces to scalar loss 2-player zero sum game.) (Use $\hat{p}_t \equiv p$ as strategy)

Blackwell's approachability theorem

A closed convex set S is approachable if and only if every halfspace containing S is approachable.



"only if" part is immediate

Let A_t = empirical average loss $= \frac{1}{t}\sum_{s=1}^{t} \ell(I_s, J_s)$

$\pi_S(A_{t-1})$ = projection onto S

$a_{t-1} = \dfrac{A_{t-1} - \pi(A_{t-1})}{\|A_{t-1} - \pi(A_{t-1})\|}$ $\qquad c_t = a_{t-1} \cdot \pi_S(A_{t-1})$

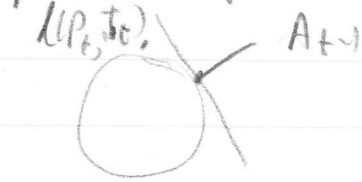
Using strategy sequence

let p_t minimize$_{p}$ $\max_j \left[a_{t-1} \cdot \bar{\ell}(p_t, j) \right]$

i.e. min max optimal for weighted loss.

Proof

We've seen all the ingredients of the proof before.



$$d(A_t, S) = \|A_t - \pi_S(A_t)\|^2$$

$$\le \|A_t - \pi_S(A_{t-1})\|^2$$

$$= \left\| \frac{t-1}{t}\left(A_{t-1} - \pi_S(A_{t-1})\right) + \frac{\ell(I_t, J_t) - \pi_S(A_{t-1})}{t} \right\|^2$$

$$= \left(\frac{t-1}{t}\right)^2 \|A_{t-1} - \pi_S(A_{t-1})\|^2 + \frac{1}{t^2}\Big\|\underbrace{\overset{\le 1}{\ell(I_t, J_t)} - \overset{\le 1}{\pi_S(A_{t-1})}}_{\le 4}\Big\|^2$$

$$+ \underbrace{\frac{t-1}{t}\left(A_{t-1} - \pi_S(A_{t-1})\right) \cdot \left(\ell(I_t, J_t) - \pi_S(A_{t-1})\right)}_{\text{would be } \le 0 \text{ if replaced by } \ell(p_t, J_t)}$$

write $\ell(I_t, J_t) = \ell(p_t, J_t)$

$$+ \underbrace{\ell(I_t, J_t) - \ell(p_t, J_t)}_{\text{bounded martingale difference } D_t}$$

$A_1 - \pi_1(A_1)$

mult by t^2

$$t^2 \|A_t - \pi_S(A_t)\|^2 \le (t-1)^2 \|A_{t-1} - \pi_S(A_{t-1})\|^2 + 4$$

$$+ 2t D_t$$

Azuma-Hoeffding + Borel Cantelli

$$\therefore\ n^2 \|A_n - \pi_n(A_n)\|^2 \le 4n + \sum_{t=1}^{n} 2t D_t$$

$$\frac{\|A_n - \pi_n(A_n)\|^2}{n^2} \le \frac{4}{n} + \frac{1}{n}\sum_{t=1}^{n} \left(\frac{2t}{n}\right) D_t \to 0 \text{ a.s.}$$

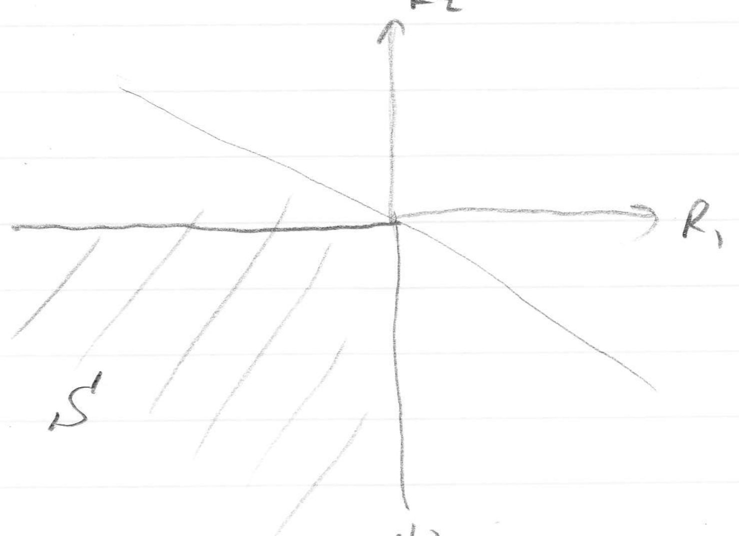
$\square$

Corollary of Blackwell's approachability thm.

Existence of Hannan consistent forecasters.

Context: Forecaster selects mixed strategies $\hat{p}_t$ for N pure strategies $\{1, \cdots, N\}$ - needs to a.s. on average do as well as each pure strategy.

So N objectives



k^{th} objective $\quad \ell^{(k)}(i, y) = \ell(i, y_t) - \ell(k, y_t)$

Apply Blackwell approachability using

$$\ell(i, y) = (\ell^{(1)}(i, y), \cdots, \ell^{(N)}(i, y))$$

Suffice to show for any nonzero a with nonnegative coordinate that the half plane $\{\underline{u} : \underline{a} \cdot u \leq 0\}$ is approachable. a is an N vector

i.e. can select p so

$$\underline{a} \cdot l(p, j) \leq 0 \quad \text{for all } j$$

$$\parallel$$

$$\sum_k a_k \{ l(p, y_t) - l(k, y_t) \} \leq 0$$ Divide a by $\sum a_i$ if necessary so a is a probability vector.

$$\parallel$$

$$\sum a_i \left(l(p, y_t) - l\left(\frac{a}{\sum a_i}, y_t\right) \right) \leq 0$$

$\therefore$ Just use $p_i = \frac{a_i}{\sum a_i}$

(You should find same predictor using quadratic potential in ps. 2)