Lecture 1

1/15/13

- Introduction

  - What is game theory?

    Multiple agents have different objectives

    Game theory offer models, language, algorithms

  - Mechanism design = inverse game theory

  - Applications in engineering

  - distributed network control - ISP-ISP interaction
    - investment in security, infrastructure

  - Online auctions/pricing (EBay, advertising auctions)

  - Spectrum sales

Biology / Social Networks / Economics

Preliminary course outline.

I. Static games of complete information
- Dominant strategies, iterated elimination of dominated strategies, mixed strategies, Nash equilibrium, saddle-point theory, rationalizability
- Evolutionarily stable equilibrium (ESS)
- Trembling hand equilibrium
- Fictitious play and convergence results
- Correlated equilibria ← need in ps 1
- Blackwell's approachability theorem

II. Dynamic games of complete information
- Subgame perfect equilibrium
- Repeated games: trigger strategies, folk theorem

III. Static games of incomplete information
- Bayes-Nash equilibrium

IV. Mechanism design and the theory of auctions
- Vikrey second price auction and its generalization (VCG mechanism)
- Revenue optimal auctions (direct revelation principle, revenue equivalence, virtual valuations (Myerson theory)
- Equilbrium bidding strategies and revenue ordering for second price and ascending auctions with correlated private information (Milgrom and Weber theory)
- Online auctions

V. Coalitions in cooperative games
- Core, Shapley value

VI. Applications: networks, Braess paradox, price of anarchy, investment in security

Example of static game

		Prisoner 2 L	Prisoner 2 R (confess)
Prisoner 1	L	1, 1	-1, 2
	R (confess)	2, -1	0, 0

prisoner gets +1 point for confessing
+1 point for going free
-1 point for serving prison time

prisoner goes free iff other prisoner does not confess.

- did demo in class with back-to-back students raising either left or right hand

- Observe: Playing R strictly dominates playing L.

If strictly dominated strategies are removed, only play (R, R) remains. That is the "solution" by iterated elimination of strictly dominated strategies

Or we could say (R,R) is an equalibrium in (strictly) dominant strategies.

Now consider variation

		Player 2 L	Player 2 R
Prisoner 1	L	1,1	-1,2
	R	2,-1	0,0
	suicide	-100,1	-100,0

Now player 2 has no dominant strategy. But R is still a dominant strategy for player 1, so if player 1 is rational and player 2 knows player 1 is rational, player 2 can eliminate the last row of the game, reducing it to previous game.

We thus again arrive at (R,R) by iterative elimination of strictly dominant strategies.

Guess $2/3$ of average game

I introduced the game of n people each selecting a number from $\{1, \cdots, 100\}$. The players with numbers closest to $2/3$ of the average split the prize money.

I asked each student to write down a number in $\{1, \cdots, 100\}$ on the questionaire. Result was an averge arround 22, $\frac{2}{3}(\text{ave}) = 15.2$, three studeqnts wrote 15 and one.

ECE568 Jan 15, 2013

Guess 2/3 of average game

Guesses	
1	
1	
1	
1	
1	
1	
2	
3	
15	Navid Aghasadeghi
15	Shaileshh Bojja Venkatakrishnan
15	James Yifei Yang
17	
20	
22	
26	
27	
34	
34	
34	
37	
40	
41	
41	
50	
50	
66	
22.8846154	Average
15.2564103	(2/3)*Average

Example of a game with solution
by weakly dominant strategies

Vikrey second price auction.

I have an object to sell. (eg pencil)

Player i has value v_i for object, $1 \le i \le n$.

(player i knows v_i).

Player i bids b_i.

Object is sold to highest bidder for the

second highest price.

Suppose you are a player, and your $v_i = \$4$

How should you bid?

discussion

Bidding truthfully is weakly dominante strategy:
- No matter what the highest bid of the other bidders, you'd do at least as well bidding \$4.
And if you don't bid \$4 you could regret it.

Example game:

Bach or Stravinsky (aka Battle of the Sexes)
(Opera vs Football)

$$\begin{array}{cc} & \text{Player 2 (prefers Stravinsky)} \\ & \begin{array}{cc} B & S \end{array} \\ \text{Player 1 (prefers Bach)} \begin{array}{c} B \\ S \end{array} & \begin{pmatrix} 3,2 & 0,0 \\ 0,0 & 2,3 \end{pmatrix} \end{array}$$

- No dominant strategies for either player.

(B,B) and (S,S) are both Nash equilibria.

Def. A vector of actions $(a_1, \cdots a_n)$ in an n-player static game is a Nash equilibrium if for each i and each alternative action a_i':
$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$$

— A "fair" resolution could be to flip a coin to randomize between the two NE's.

Heads — both see Bach
Tails — both see Stravinsky

Are there any Nash equilibrium in (independently) randomized strategies?

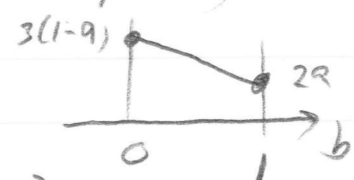
Player 1 using probability dist^n $(a, 1-a)$
Player 2 " " " $(b, 1-b)$

((1,0) is a degenerate mixed strategy), $((1,0),(1,0))$ and $((0,1),(0,1))$ are both Nash equilibria. Any others?

Suppose $0 < a < 1$. and $0 < b < 1$. (Can't have NEP with only one mixed strategy for this game.)

[If player 2 chooses B her average payoff is $2a$
" " " " S " " " " $3(1-a)$

In order for $0 < b < 1$ to be a best response, these two must be equal.



(To see it, $\max_b \{(2a)b + (3(1-a))(1-b)\}$ is achieve in the interior iff $3(1-a) = 2a$)

$2a = 3(1-a) \Rightarrow 5a = 3 \qquad a = 3/5.$

$$\begin{array}{ccc} & 3/5 & 2/5 \\ 3/5 & \left(\frac{6}{25}\right. & \left.\frac{4}{25}\right) \\ 2/5 & \left(\frac{9}{25}\right. & \left.\frac{6}{25}\right) \end{array}$$

By symmetry, $b = 3/5$.

So $\left(\left(\frac{3}{5}, \frac{2}{5}\right)\left(\frac{2}{5}, \frac{3}{5}\right)\right)$ is the unique nondegenerate <u>mixed</u> Nash equilibrium.

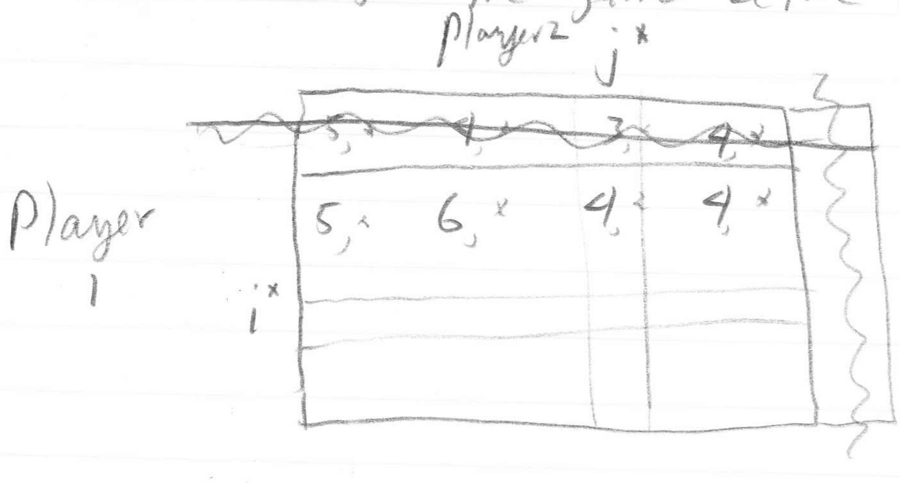
Average payoffs $\left(3 \cdot \frac{6}{25} + 2 \cdot \frac{6}{25} = \frac{6}{5}\right.$ $\left(\frac{6}{5}, \frac{6}{5}\right)$ worse than $(2,3)$ or $(3,2)$.

Now that we've seen what a NE is, let us return to the "Guess 2/3 of average" game. Clearly $(1, 1, \dots, 1)$ is a Nash equilibrium.

The following is in the statement of problem 1:

<u>Claim</u> If iterated elimination of weakly dominated strategies (IEWDS) leaves only one strategy pair remaining, it is a NE.

Follows from: If after one step of IEWDS there exists an NE (i^*, j^*), then (i^*, j^*) is also an NE for the game before the step.



- with dominated lines added back in, i^* is still a best response to j^*.

Example game

Rock-Paper-Scissers

		Player 2 R	P	S
Player 1	R	0,0	-1,1	1,-1
	P	1,-1	0,0	-1,1
	S	-1,1	1,-1	0,0

Do demo.
Paper beats rock
Rock beats scissers
Scissers beats paper

- No strategy is dominatated (even weakly) so iterated elimination of dominated strategies doesn't work.

Consider mixed strategies (p, q)
(we think of probability distributions on finite sets as row vectors)

Nash equilibrium: $\left((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \right)$

This is a 2-player zero-sum game.
(we'll see all NEP's have the same value for such games. If either player used any other strategy, the other players best response would have a strictly positive value. Hence, NEP is unique.)

Matching Pennys Game:

		Player 2 (wants different) Heads b	Tails $1-b$
Player 1 (wants to match)	Heads a	$+1 \quad -1$	$-1 \quad +1$
	Tails $1-a$	$-1 \quad +1$	$+1 \quad -1$

(2-player zero sum)

or $A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

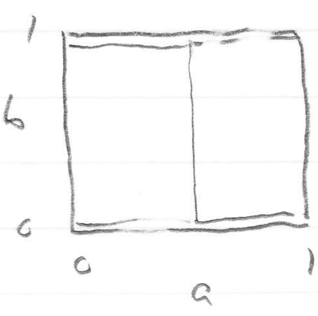
- no dominant strategies
- no Nash equilibrium in pure strategies.

Seek mixed strategy NE: $\big((a, 1-a), (b, 1-b)\big)$

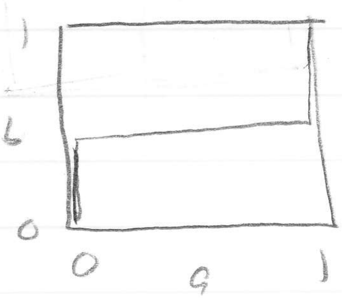
- There are none with $a \in \{0,1\}$ or $b \in \{0,1\}$.
- To be an NE with $0<b<1$, need both responses of player 2 to give same mean payoff.

Need $-a + (1-a) = a - (1-a) \Leftrightarrow a = \frac{1}{2}$

Equivalently

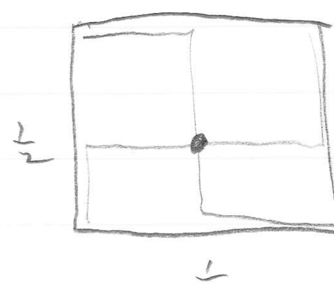


Best b for given a



Best a for given b

Intersection



So $(\frac{1}{2}, \frac{1}{2})$ is the unique NE

Finding a mixed NEP (we see its basically a combinatorial problem. Finding supports)

Consider a general $m \times n$ two player matrix game.

p = dist^n for player 1

q = " " " 2

player 1 is concerned with $\max\limits_{p} \; p A_1 q^T$

player 2 " " " $\max\limits_{q} \; q A_2 p$.

Suppose $S_1 = \{i : p_i > 0\}$, $S_2 = \{j : q_j > 0\}$

$|S_1| = n_1$ $\quad S_2 = |n_2|$

For NE need:

$(pA_2)_{(j)}$ all equal for $j \in S_2$ ($n_2 - 1$ equations) ← call value V_1

$(pA_2)_{(j)} \geq V_2$ for all $j \in S_2^c$

$(A_1 q)_{(i)}$ all equal for $i \in S_1$ ($n_1 - 1$ equations) ← call value V_1

$(A_1 q)_{(i)} \geq V_1$ for all $i \in S_1^c$

$\therefore$ p, q satisfy $(n_1 - 1) + (n_2 - 1)$ equality constraints

plus, $\sum\limits_{S_1} p_i = 1$, $\sum\limits_{S_2} q_j = 1$ $\quad \therefore$ $n_1 + n_2$ constraints for $n_1 + n_2$ probabilities.

Lecture 3
1/22/2013

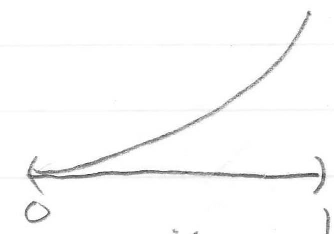
Existence of Nash equilibria

- We'll see that a Nash equilibrium exists (in mixed strategies) in great generality. (Today show for finite games)

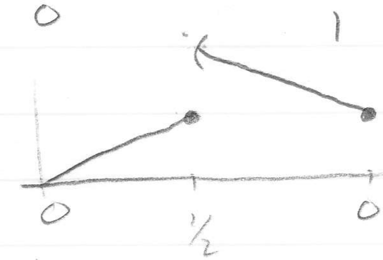
- A special case is a one player game. Player wants to select x to maximize a function $f: S \to \mathbb{R}$

e.g. $S = (0,1)$ $\quad f(x) = x^2$

S not closed



$S = [0,1]$ $\quad f(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ \frac{3}{2} - x & \frac{1}{2} < x \le 1 \end{cases}$



- not (upper semi) continuous.

$S = [0, \infty]$, $f(x) = \frac{1}{1+x}$

A not bounded (bounded + closed = compact (sequences have limit point))

[Weirstrass extreme value theorem
f continuous, S (sequentially) compact
$\Rightarrow f$ obtains maximum.]

[Image $f(S)$ is compact]

Proof. Let $V = \sup_{x \in A} f(x) \le +\infty$. Let $V_n \uparrow V$. Let $f(x_n) \ge V_n$. S compact so subsequence x_{n_k} converges to some x_*. By continuity, $V \ge f(x_*) = \lim_{k \to \infty} f(x_{n_k}) \ge V_n$ for all n. $\therefore f(x^*) = V$

/

<u>Definition</u> A <u>strategic form game</u> is given by a triplet $(I, (S_i : i \in I), (u_i : i \in I))$

where I = set of players (finite)

S_i = set of strategies/actions of player i

(Note assumption $\times$ is a <u>product</u> set)

$u_i : S \to \mathbb{R}$ where $S = \underset{i \in I}{\times} S_i$ $(S_1 \times S_2 \times \cdots \times S_n)$ if $I = \{1, \dots, n\}$

let's assume this is

<u>Useful notation</u> If $x \in S$ we often write $x = (x_i, x_{-i})$ where x_i is the i^{th} coordinate of x and x_{-i} represents all the other coordinates.

For example, if $I = \{1, 2, 3, 4\}$ and

$x = (5, 6, 3, 2)$ then $x_2 = 6$ and $x_{-2} = (5, 3, 2)$

$$\underset{x_1}{5}\ \underset{x_2}{6}\ \underset{x_3}{3}\ \underset{x_4}{2}$$

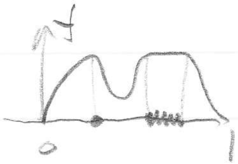
so $x = (x_2 = 6, x_{-2} = (5, 3, 2))$

This is a useful abuse of notation.

It wouldn't make sense to just write $x = (6, (5, 3, 2))$.

In game theory we often encounter set valued functions

Def Let $f : S \to \mathbb{R}$. Then



$$\arg\max_{x \in S} f = \{x \in S : f(x) = \sup_{y \in S} f(y)\}$$

Example The best response functions

$$B_i(x_{-i}) \triangleq \arg\max_{x_i} u_i(x_i, x_{-i})$$

<u>Def</u> x^* is a NE if for each i and each $x_i' \in S_i$,
$$u_i(x^*) \geq u_i(x_i', x_{-i}^*).$$

Equivalently,
$$x_i^* \in B_i(x_{-i}^*) \text{ for all } i.$$

Equivalently, grouping together,
$$x^* \text{ is an NE iff } x^* \in B(x^*)$$

where $B(x) = (B_1(x_{-1}), B_2(x_{-2}), \cdots, B(x_{-n}))$

$\therefore$ NE's are fixed points of the function comprised of best response functions.

Existence of NE follows from fixed point theorems.

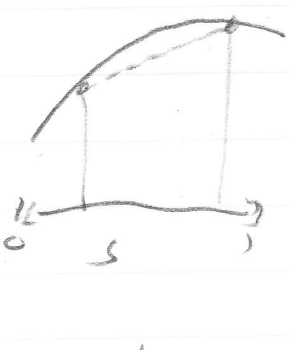
Review

* S, $S \subset \mathbb{R}^n$, is convex if

$$x, y \in S \Rightarrow \lambda x + (1-\lambda) y \in S \quad \text{for any } \lambda \in [0,1]$$

* $f: S \to \mathbb{R}$ is a concave function ($\cap$)

if $\quad f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y) \quad$ (not identically $+\infty$)



$\cup$

(convex functions = $-$(concave functi)

Another useful notation: (for mixed strategies)

If $f : S \to \mathbb{R}$ and σ is a probability distribution on S, then

$$f(\sigma) = \sum \sigma_i f_i$$

or

$$\int_S f(s)\, d\sigma(s)$$

When extended to mixed strategies, $\underset{\sigma \in \Delta(A)}{\arg\min}\, f(\sigma)$ is the set of probability measures giving probability one to $\underset{x}{\arg\min}\, f(x)$.

This set is thus a convex set of probability measures.

Bauer fixed point theorem

(for proofs see Economics for Mathematicians, Cassels)

Let S be a simplex or unit ball in $\mathbb{R}^n$ and let $f: S \to S$ be continuous. Then there exists $x^* \in S$ so that $f(x^*) = x^*$.

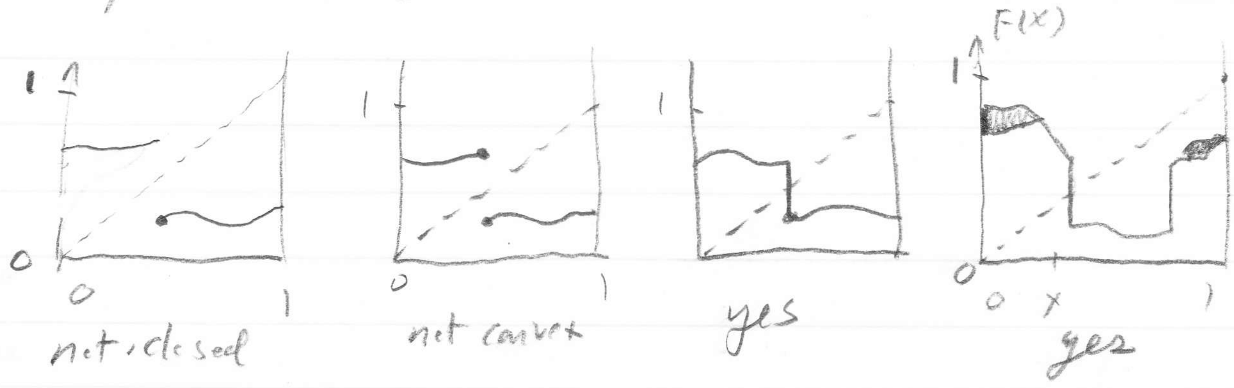
- point to point mapping: we need a point to set mapping.

Kakutani Let $f: A \rightrightarrows A$ be a set valued correspondence. (i.e. for $x \in A$, $f(x) \subset A$) such that

1. A is compact, convex, nonempty subset of $\mathbb{R}^n$, for some n.
2. $f(x)$ is nonempty for each x.
3. $f(x)$ is a convex set for each x.
4. The graph of f, ($\{(x, y) : x \in A, y \in f(x)\} \subset A \times A$) is closed.

Then there exist x^* so that $x^* \in F(x^*)$.

Examples $A = [0,1]$



<u>Existence for finite games (Nash)</u> Let

$(I, (S_i : i \in I), (u_i : i \in I))$ be a finite game

(i.e. I, and sets S_i are all finite) Then there

exists an NE of <u>mixed</u> strategies.

<u>Proof</u> Let Σ_i = set of probability vectors for S_i
(i.e. mixed strategies for player i)

$$B_i(\sigma_{-i}) \stackrel{\Delta}{=} \arg\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})$$

$$\Sigma = X_i \Sigma_i \qquad (\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma)$$

$$B(\sigma) = (B_1(\sigma_1), \ldots, B_n(\sigma))$$

Then σ is an NE iff $\sigma \in B(\sigma)$.

Check that $B(\sigma)$ satisfies the condition of

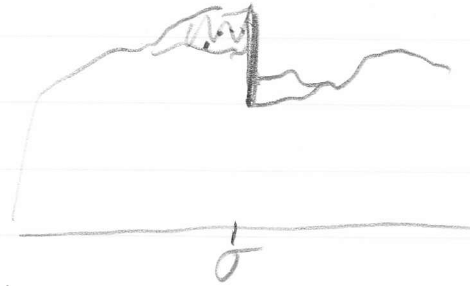
the Kakutani fixed point theorem.

(Finish up on proof of Nash thm)

Lecture 4

1/24/13

Add check that graph of $B(\sigma)$ is closed.



(Review: If $x_i \geq 0$, $x_i \to x_\infty$ then $x_\infty \geq 0$. $[0,\infty)$ is closed)

Need to show:

Need: If $\sigma^{(n)} \to \sigma^{(\infty)}$

and $\hat{\sigma}^{(n)} \in B(\sigma^{(n)})$ such that $\hat{\sigma}^{(n)} \to \hat{\sigma}^{(\infty)}$

Then $\sigma^{(\infty)} \in B(\sigma^{(\infty)})$.

Reduces to a question for each $i \in I$

or: If $\sigma_{-i}^{(n)} \to \sigma_{-i}^{(\infty)}$

$\hat{\sigma}_i^{(n)} \in B_i(\sigma_{-i}^{(n)})$ such that $\hat{\sigma}_i^{(n)} \to \hat{\sigma}_i^{(\infty)}$

then $\hat{\sigma}_i^{(\infty)} \in B_i(\sigma_i^{(n)})$

Let σ_i' be any strategy in Σ_i. Then

$$u_i(\hat{\sigma}_i^{(n)}, \sigma_{-i}^{(n)}) \geq u_i(\sigma_i', \sigma_{-i}^{(n)}) \text{ for all } n.$$

$$\downarrow \qquad\qquad \downarrow$$

$$u_i(\hat{\sigma}_i^{(\infty)}, \sigma_{-i}^{(\infty)}) \geq u_i(\sigma_i', \sigma_{-i}^{(\infty)})$$

$x_i \geq y_i$ $\quad x_i \to x^\infty$ $\quad y_i \to y^\infty$ then $x_i^{(\infty)} \geq y^{(\infty)}$.

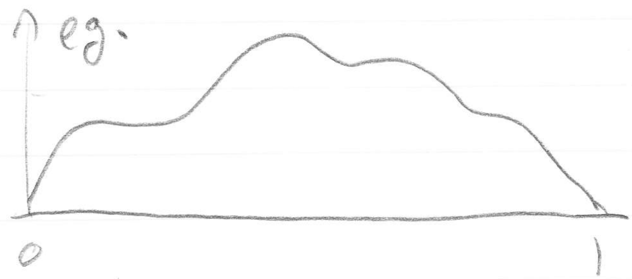
Games with infinite pure strategy sets

(we'll) see a sufficient cond. for pure strategy NE.

Suppose $C \subset \mathbb{R}^n$, C convex

<u>Def</u> f is quasi-concave if the t-upper level sets of f: $L_f(t) \triangleq \{x : f(x) \geq t\}$ are all convex sets.

e.g.



<u>Theorem</u> (Debreu, Glicksberg, and Fan) Consider a strategic form game $(I, (S_i), (u_i))$ with I finite such that

1. S_i is nonempty, compact, convex subset of $\mathbb{R}^{n_i}$
2. $u_i(x)$ is continuous on $S = S_1 \times \cdots \times S_n$
3. $u_i(s_i, s_{-i})$ is quasiconcave in s_i

Then the game has a pure strategy NE.

(<u>Note</u>: This generalizes Nash's thm about finite games mixed strat.)

Note The allocation game of problem 2

ps 1 is covered here (if we also assume

$u_i(0)$ is finite.)

Proof As for Nash's theorem for finite games, apply Kakutani theorem. Need existence of a fixed point for the correspondence $B : S \rightrightarrows S$

$$B(s) = B_1(s_{-1}) \times \cdots \times B_n(s_{-n})$$

1. S nonempty compact, convex ✓
2. $B(s) \neq \emptyset$ (Weirstrass) ✓
3. $B(s)$ is a convex set for each s ✓
4. $\mathcal{G}_B = \{(s,t) : t \in B(s)\}$ is a closed subset of $S \times S$

(Note: It can be shown that continuous games with compact strategies have mixed NE's. I may cover it in class. Proof is by approximation by finite games. "Glickberg Theorem.

Example Zero sum game, two points on circle. One player wants to min distance, other to maximize it.

Saddlepoints and 2-player zero-sum games (Von Neuman 1928)

A two player game is a zero sum 2-player game if the sum of the payoffs is zero.

For such games, we can let $l(p,q)$ denote the payoff for player 2, where p is the strategy of player 1 and q is the strategy of player 2.

The payoff function for player 1 is $l_1(p,q) = -l_2(p,q)$.
Equivalently,
Player 1 seeks to select p to minimizes $l(p,q)$ and player 2 seeks to maximize it.

An NEP in this context is called a saddlepoint.

Fact 1 If p, q each range over compact, convex sets and $l(\cdot,\cdot)$ is jointly continuous, quasiconvex in p and (quasi) concave in q, a saddlepoint exists. (e.g. $l(p,q) = pAq^T$ for dist^n p,q)

Proof: [Debreau, Glicksberg, Fan] thm based on fixed points. $\square$

<u>Fact 2</u> For any function $l(p,q) \to \mathbb{R}$

$$\sup_q \inf_p l(p,q) \le \inf_p \sup_q l(p,q) \qquad (*)$$

<u>Note</u> If player 1 does $\inf p$ and player 2 $\sup q$, $(*)$ means player choosing second (after first player) has the advantage.

<u>Proof</u> wlog assume $\inf_p \sup l(p,q) \in [-\infty, +\infty)$. Fix any finite constant c with $\inf_p \sup_q l(p,q) < c$. By def^{n} of $\inf_p$, there exists $\bar{p}$ so that

$$\sup_q l(\bar{p}, q) \le c$$

So $$\sup_q \inf_p l(p,q) \le \sup_q l(\bar{p}, q) \le c$$

Since c is arbitrary, $(*)$ follows.

(we use that fact that if $x \in [-\infty, \infty)$ and if $y \le c$ for any $c > x$, then $y \le x$) $\square$

<u>Definition</u> The number $\Delta = \inf_p \sup_q l(p,q) - \sup_q \inf_p l(p,q)$ is called the <u>duality gap</u>.

Fact 3 If a saddlepoint exists, then the duality gap is zero i.e.

$$\inf_p \sup_q l(p,q) = \sup_q \inf_p l(p,q) \qquad (1)$$

Proof If $(\bar{p},\bar{q})$ is a saddlepoint, then by definition,

$$\inf_p l(p,\bar{q}) = l(\bar{p},\bar{q}) = \sup_q l(\bar{p},q). \qquad (2)$$

Let V denote the common value of the quantities in (2). Then each side of (1) is easily seen to lie in the interval $[\inf_p l(p,\bar{q}), \sup_q l(\bar{p},q)]$ and hence are both equal to V. $\square$

Thus, a minmax problem with a saddlepoint has a value V equal to either side of (1) and $l(\bar{p},\bar{q}) = V$ for all saddle points $(\bar{p},\bar{q})$.

Def. $\bar{p}$ is minmax optimal if $\sup_q \ell(\bar{p}, q) = \inf_p \sup_q \ell(p,q)$ and $\bar{q}$ is maxmin optimal if $\inf_p \ell(p, \bar{q}) = \sup_q \inf_p \ell(p,q)$.

Fact 4 A pair $(\bar{p}, \bar{q})$ is a saddlepoint if and only if $\bar{p}$ is minmax optimal, $\bar{q}$ is maxmin optimal, and there is no duality game.

Proof. (only if) The proof of Fact 3 shows that if $(\bar{p}, \bar{q})$ is a saddlepoint then $\bar{p}$ is minmax optimal, $\bar{q}$ is max-min optimal, and duality gap is zero.

(if) Suppose $\bar{p}$ is minmax optimal, there is no duality gap, and $\bar{q}$ is maxmin optimal. Using these properties in the order just listed yields:

$$\sup_q \ell(\bar{p}, q) = \inf_p \sup_q \ell(p,q) = \sup_q \inf_p \ell(p,q) = \inf_p \ell(p, \bar{q}).$$

Therefore

$$\ell(\bar{p}, \bar{q}) \le \sup_q \ell(\bar{p}, q) = \inf_p \ell(p, \bar{q}) \text{ and } \ell(\bar{p}, \bar{q}) \ge \inf_p \ell(p, \bar{q}) = \sup_q \ell(\bar{p}, q)$$

Thus, $(\bar{p}, \bar{q})$ is a saddlepoint. $\square$

<u>Fact 5</u> For bilinear two player zero sum games ($\ell(p,q) = pAq^T$) the minmax problem for player 1 and maxmin problem for player 2 are dual linear programs.

min max problem: $\min_{x \in \Delta} \max_{y \in \Delta} x A y^T$

Same as $= \min_{x,t} t$ $\quad xA \le te^T$ $\quad x \geq 0$ $\quad xe = 1$

$e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$= \min_{\substack{x,t \\ x \geq 0}} \max_{\substack{\lambda \geq 0 \\ \mu}} t + (xA - te^T)\lambda + \mu(1 - xe)$$

(key step in deriving dual)

$$= \max_{\substack{\lambda \geq 0 \\ \mu}} \min_{\substack{x,t \\ x \geq 0}} \mu + x(A\lambda - \mu e) + t(1 - \lambda e)$$

$$= \max_{(\lambda,\mu):\ \lambda \geq 0} \mu$$

$A\lambda \geq \mu e$

$\lambda e = 1$

Replace λ by λ^T.

$$= \max_{\lambda \in \Delta} \min_{x} xA\lambda^T$$

Δ = set of probability distributions of appropriate dimension

<u>Note</u> key step is usually proved using Farkas lemma, which is dual (dual (cone span)) = cone span

$a_{ij} x_j$

An example of duality in linear programing:

$$\begin{pmatrix} \max \sum_j p_j x_j \\ Ax \leq c \\ x \geq 0 \end{pmatrix}$$

c_i = supply of resource i

a_{ij} = amount of resource i going into one unit of j

p_j = sale price per unit of j produced

<u>Dual</u>

$$= \max_{x \geq 0} \min_{\mu \geq 0} p^T x + \mu^T (c - Ax)$$

$$= \min_{\mu \geq 0} \max_{x \geq 0} \mu^T c + (p^T - \mu^T A) x$$

$$= \min_{\mu \geq 0} \mu^T c \qquad \sum_i \mu_i c_i$$

$$\mu^T A \geq p^T \qquad \sum_i \mu_i a_{ij} \geq p_j$$

p_j is "market rate".

Someone wants to buy inventory.

Selects prices per commodity to "explain" the market prices.

- μ_i is price seller could attach to resource i so that production cost of each unit of j is at least p_j.

Correlated Equilibria

(Concept due to Aumann (1974))

Example:

	D (Dove)	H (Hawk)
D	4,4	1,5
H	5,1	0,0

variation of "chicken game"
Hawk → dare
Dove - chicken

Hawk = "aggressive behavior"
Dove = "passive behavior"

No dominated strategies

(D, H) and (H, D) are pure strategy NE.
$((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is a mixed strategy NE with payoff (2.5, 2.5)

Better to randomize between payoffs (5, 1) and (1, 5), using public coin flip. ⇒ payoffs (3, 3).

If (0,0) replaced by (2,2) get prisoner's dilemma.

Note - public randomization can only yield convexification of set of NE payoff vectors.

Correlated private random signals can yield additional equilibrium.

There are 2 definitions in literature (equivalent).

Simplest is as follows. Begin with a joint distribution p over $S_1 \times S_2$ (strategy pairs).

Imagine a coordinator selecting $(i,j) \in S_1 \times S_2$ at random using p. Then coordinator tells player 1 to play i, player 2 to play j (but does not reveal j to player 1 or i to player 2.)

Then p is called a <u>correlated equilibrium</u> if no player has an incentive to unilaterally disobey the coordinator.

<u>Claim</u>: For this game $\begin{array}{c|cc} & D & H \\ \hline D & \tfrac{1}{3} & \tfrac{1}{3} \\ H & \tfrac{1}{3} & 0 \end{array}$ is a correlated equilibrium.

- Given player 1 is told to play H, he knows player 2 was told to play D, so player 1 should indeed play H.

- Given play 1 is told to play D, by Bayes rule, the conditional distn of play of player 2 is D and H with equal probability. So player 1 has mean return 2.5 for either D or H. No incentive to deviate

Similarly, player 2 has no incentive to deviate

Under this equilibrium, payoff vector is

$\left(\frac{10}{3}, \frac{10}{3}\right)$ (beats convex combination of NE).

- For this game, mixing NE points,
  coordinator says "player 1 play D, player 2 H"
  or " " " H, player 2 D

If coordinator ever publically told both to play 'D', then each player would have an incentive to deviate

However, point is coordinator's signals are private. Sometimes coordinator tells both players to play D, but players don't know exactly when that happens.

In general, a correlated equilibrium is a probability distribution p on $S_1 \times \cdots \times S_n$ such that for any player i and any action s_i for player i with positive marginal probability, and any alternative action s_i': (prime here only)

$$(s_i \to s_i') \qquad \sum_{s_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p(s_i; s_{-i}) u_i(s_i', s_{-i})$$

(i.e. player i has no incentive to deviate when told to play s_i.

Note 1: If both sides are divided by the marginal probability of s_i, can write as

$$\sum_{s_{-i}} p(s_{-i} | s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p(s_{-i} | s_i) u_i(s_i', s_{-i})$$

Note 2: Alternative definition of correlated equilibrium is with respect to an information structure $(\Omega, \{\mathcal{H}_i\}, p)$ $\mathcal{H}_i$'s are σ-algebras, Ω finite. Player i's strategy S_i should be $\mathcal{H}_i$-measurable. All that matters for payoffs is $(s_1, \ldots, s_n)$ joint pdf, and revealing minimum info to player i gives least incentive to deviate.