University of Illinois **Spring 2013**

ECE 586BH: Problem Set 6: Problems and Solutions
Revenue optimal selling mechanisms, auctions with interdependent values

Due: Thursday, April 25 at beginning of class
Reading: V. Krishna, *Auction Theory,* Chapter 5 (based largely on Myerson 1982 paper) and Chapter 6 (based on Milgrom and Weber 1983 paper)

1. **[VCG example - sale of two items]**
Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a, b\}$ to three buyers. A buyer can buy none, one, or both of the objects. For simplicity, assume the valuation function of each buyer depends only on the set of objects assigned to that buyer. The values are:

$$\begin{aligned} u_1(\emptyset) = 0,\ u_1(\{a\}) = 10,\ u_1(\{b\}) = 3,\ u_1(\{a, b\}) = 13 \\ u_2(\emptyset) = 0,\ u_2(\{a\}) = 2,\ \ u_2(\{b\}) = 8,\ u_2(\{a, b\}) = 10 \\ u_3(\emptyset) = 0,\ u_3(\{a\}) = 3,\ \ u_3(\{b\}) = 2,\ u_3(\{a, b\}) = 14 \end{aligned}$$

To be definite, suppose the standard version of the payment rule is used whereby for each buyer i, $m_i(\widehat{u})$ is the maximum welfare of the other buyers minus the realized welfare of the other buyers, both computed using the reported valuation functions.

(a) Determine the assignment of objects to buyers and the payments of the buyers, under truthful bidding.

Solution: If the two items are awarded to the same buyer, the maximum welfare is 14. A larger welfare is achieved by awarding the items to different buyers, and the maximum of 18 is achieved when buyer 1 is assigned object a, buyer 2 is assigned object b, and buyer 3 is assigned no object. To determine m_1, note that the maximum welfare for buyers 2 and 3 alone is 14 (obtained by assigning the object to buyer 3) and the realized welfare for buyers 2 and 3 is 8. So $m_1 = 14 - 8 = 6$. To determine m_2, note that the maximum welfare for buyers 1 and 3 alone is also 14, and the realized welfare for buyers 1 and 3 is 10. So $m_2 = 14 - 10 = 4$. Finally, $m_3 = 0$. In summary, buyer 1 gets object a for payment 6, buyer 2 gets object b for payment 4, and buyer 3 gets no object and makes no payment.

(b) Discuss why buyer 3 might have an objection to the outcome.

Solution: Buyer 3 could point out (and maybe even file a lawsuit) that he bid 14 for the pair $\{a, b\}$ and lost, while the mechanism sold $\{a, b\}$ for a total payment of 4+6=10. (This is an example of a problem auctioneers have addressed in some instances by using *core selecting auctions,* for which the VCG price vector is projected onto the set of price vectors such that no buyer could claim discrimination. A problem with this fix is that the projection destroys the incentive compatibility property of the VCG mechanism.)

2. **[About the virtual valuation functions in revenue optimal seller mechanisms]**
Given a pdf f_i with support equal to the interval $[0, \omega_i]$, the virtual valuation function is defined on the same interval and is given by $\psi_i(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$.

(a) Show that $E[\psi_i(X_i)] = 0$.

Solution: $E[\psi_i(X_i)] = \int_0^{\omega_i}\left(x_i - \frac{1-F_i(x_i)}{f_i(x_i)}\right) f_i(x_i)dx_i = E[X_i] - \int_0^{\omega_i}(1-F_i(x_i))dx_i = 0$, where we use the integration by parts formula (aka area rule) for expectations of nonnegative random variables: $E[X_i] = \int_0^{\infty}(1-F_{X_i}(x_i))dx_i$.

(b) Find ψ_i in case X_i is exponentially distributed with parameter $\lambda > 0$.

Solution: $\psi_i(x_i) = x_i - \frac{1-(1-e^{-\lambda x_i})}{\lambda e^{-\lambda x_i}} = x_i - \frac{1}{\lambda}$.

(c) Suppose ψ_i is an increasing function on $[0, \omega_i]$. Work backwards to express the CDF F_i in terms of ψ_i, being as explicit as possible. To be definite, suppose $F_i(0) = 0$. What additional assumptions are needed on ψ so that F_i is nondecreasing with $F(\omega_i) = 1$?

Solution: The definition of ψ_i can be rearranged to get $\frac{f_i}{1-F_i} = \frac{1}{x_i - \psi_i}$. If we define y to be the function $y(x_i) = \ln(1 - F_i(x_i))$ for $0 \le x_i \le \omega_i$, then the equation becomes $y_i' = -\frac{1}{x_i - \psi_i}$. With the initial condition $y(0) = 0$, this yields

$$y(x_i) = -\int_0^{x_i} \frac{1}{t_i - \psi_i(t_i)} dt_i$$

and therefore

$$F_i(x_i) = 1 - \exp\left(-\int_0^{x_i} \frac{1}{t_i - \psi_i(t_i)} dt_i\right)$$

In order for F to be nondecreasing with $F_i(\omega_i) = 1$ we need $\psi_i(t_i) < t_i$ for $0 \le i < \omega_i$ and $\int_0^{\omega_i} \frac{1}{t_i - \psi_i(t_i)} dt_i = \infty$.

3. **[A mechanism to maximize a mixed objective function]**

Suppose N buyers have known independent valuations with the valuation of buyer i having strictly positive pdf f_i over the interval $[0, \omega_i]$. As discussed in class, the revelation principle and revenue equivalence principle can be used to help identify a simple structure for the (revenue) optimal auction. The revelation principle can also be used to help derive a maximum welfare auction; the auction turns out to be the second price auction. For this problem, suppose α is fixed with $0 \le \alpha \le 1$. Find a direct mechanism (Q, M) with truthful equilibrium that is individually rational, which, at the equilibrium point, maximizes $\alpha \times$(revenue to seller) + $(1-\alpha) \times$ (social welfare). By the revelation principle, there is no loss of optimality in restricting attention to such mechanisms. (Hint: Follow the derivation of the revenue optimal auction.)

(a) Does the revenue equivalence principle apply? If so, what can be deduced from it?

Solution: Yes, the revenue equivalence principle requires only that the allocation probabilities q_i are nondecreasing, and then yields, for an IC mechanism,

$$m_i(x_i) = m_i(0) + x_i q_i(x_i) - \int_0^{x_i} q_i(t_i)dt_i$$

For the sake of revenue maximization, we take $m_i(0) = 0$. (The choice of $m_i(0)$ has no effect on the welfare achieved by the mechanism.) As in the case for revenue optimal auctions, $E[m_i(X_i)] = E[\psi_i(X_i)I_{\{i \text{ wins}\}}]$, where ψ_i is the virtual valuation function used for revenue optimal auctions: $\psi_i(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$.

(b) Propose a selection rule for the auction.

Solution: The objective to be maximized is

$$E\left[\sum_{i=1}^{N} \alpha\psi(X_i)I_{\{i \text{ wins}\}} + (1-\alpha)X_i\right],$$

or equivalently,

$$E\left[\sum_{i=1}^{N}\psi_i^{(\alpha)}(X_i)I_{\{i \text{ wins}\}}\right] \tag{1}$$

where $\psi^{(\alpha)}(x_i) = x_i - \frac{\alpha_i(1-F_i(x_i))}{f_i(x_i)}$. The selection rule that maximizes the objective in (1) is to select the winner from among those buyers in the set $\arg\max_i \psi_i^{(\alpha)}(X_i)$ if $\max_i \psi_i^{(\alpha)}(X_i) \geq 0$, and to select no winner otherwise.

(c) State a regularity assumption similar to the one made about the functions ψ_i for the revenue optimal auction, which insures the selection rule proposed in part (b) is optimal for the mixed objective function of this problem. Finally, identify the corresponding payment rule, that reduces to the second price payment rule in the special case $\alpha = 0$.

Solution: The optimality of the selection rule given in the solution to (b) above is predicated on the selection probabilities $q_i(x_i)$ being nondecreasing in x_i. This will be the case under the following regularity assumption: the function $\psi^{(\alpha)}$ is increasing over the interval $[0, \omega_i]$ for $1 \leq i \leq N$. (This assumption is less restrictive for smaller α's.) The payment rule has the same form as a payment rule for the optimal auction, but with ψ replaced by $\psi^{(\alpha)}$. That is, defining

$$y_i^{(\alpha)}(x_{-i}) = \min\{z_i : \psi_i^{(\alpha)}(z_i) \geq 0 \text{ and } \psi_i^{(\alpha)}(z_i) \geq \psi_j^{(\alpha)}(x_j), j \neq i\},$$

the payment rule is given by $M_i(x) = y_i^{(\alpha)}(x_{-i})I_{\{i \text{ wins}\}}$.

4. **[On some properties of (revenue) optimal auctions]**
Consider the basic revenue optimal direct mechanism with truthful equilibrium, assuming N buyers with independent valuations, with the valuation of buyer i known to have pdf f_i with support equal to the interval $[0, \omega_i]$, for each i. Assume regularity, i.e. the virtual payoff functions ψ are strictly increasing.

(a) The function $m_i(x_i)$ determines only the expected payment of buyer i, given buyer i reports x_i. A particular form of the actual payment is $M_i(x_i) = y_i(x_{-i})I_{\{i \text{ wins}\}}$, where

$$y_i(x_{-i}) = \min\{z_i : \psi_i(z_i) \geq 0 \text{ and } \psi_i(z_i) \geq \psi_j(x_j), j \neq i\},$$

Truthful reporting is a Bayes-Nash equilibrium. Is truthful reporting for a given buyer i also a weakly dominant strategy? Explain your answer.

Solution: Yes. Fix a player i and let $W_i = y_i(\widehat{X}_{-i})$, where $\widehat{X}_{-i}$ denotes a vector of possible bids of the other players, who are not necessarily reporting their true values. If player i bids $\widehat{X}_i$, he wins if $\widehat{X}_i > W_i$, he loses if $\widehat{X}_i < W_i$, and he may or may not win if $\widehat{X}_i = W_i$. If he wins he pays W_i and if he loses he pays nothing. Thus, from the perspective of player i, the game looks identical to a second price auction, with W_i representing the highest bid of the other players. By the theory of second price auctions, bidding truthfully for buyer i is a weakly dominant strategy. (A proof of this property is as follows. Consider the case $W_i = w_i$ and $X_i = x_i$. If $x_i < w_i$ then player i would not want to win and pay w_i, so bidding x_i is a best response. If $x_i > w_i$ then buyer i would be better off winning and paying w_i, which happens if the player bids truthfully. If $w_i = x_i$ the payoff of the buyer is zero for any bid. So in all three cases, bidding truthfully is a best response. So no matter what the other players bid, it is a best response for buyer i to bid truthfully.)

(b) For the payment rule in part (a), is it possible that for buyer i and some realization of the vector X, buyer i bids truthfully and is a winner, but his payment is strictly larger than X_i?

Solution: No. Player i wins only if $X_i \geq W_i$, and W_i is the payment if player i wins.

(c) Another form for the payment rule is $M_i(x) = P_i(x_i)I_{\{i \text{ wins}\}}$, where the function P_i is selected so that $E[M_i(X)|X_i = x_i] = m_i(x_i)$, where m_i is the same function as before, determined by incentive compatibility. This rule has the advantage for player i that if he wins, he will know what his payment will be, no matter what the other buyers bid. Express the function P_i in terms of the function q_i. (Truthful reporting should be a Bayes-Nash equilibrium.) Is truthful reporting a weakly dominate strategy for each buyer?

Solution: The mean payment for bid x_i by player i, assuming truthful reporting by the other players, is $q_i(x_i)P_i(x_i)$, which must equal $m_i(x_i)$. Therefore, $P_i(x_i) = \frac{m_i(x_i)}{q_i(x_i)}$. By revenue equivalence (and taking $m_i(0) = 0$ for revenue optimality), $m_i(x_i) = q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i$ so that

$$P_i(x_i) = x_i - \frac{\int_0^{x_i} q_i(t_i)dt_i}{q_i(x_i)}.$$

Truthful reporting is not a weakly dominate strategy for this payment rule. If the other players don't report truthfully, then $q_i(x_i)$ no longer must be the probability player i wins for bid x_i. For example, if all the other players bid zero, then player i can win with probability one by bidding his reserve price, $r_i = \min\{z_i : \psi(z_i) = 0\}$ (or slightly more) and pay $P_i(r_i) = r_i$. That would be a stricly better response then bidding truthfully if X_i is large.

5. **[Illustration of need for affiliation assumption]**
A random vector $X = (X_1, X_2, X_3, X_4)$ is generated as follows. A fair coin is flipped, and if heads shows, $X = \Pi(1, 0.9, 0.9, 0.9)$, and if tails shows, $X = \Pi(1, 1.3, 0, 0)$, where Π is a random permutation acting on 4-tuples, with all 4! possibilities having equal probability. Let $V = X_1 + X_2 + X_3 + X_4$. Note that the distribution of X is symmetric and V is a symmetric increasing function of X. Find and compare $E[V|X_1 = 1]$ and $E[V|X_1 = 1, Y_1 < 1]$, where $Y_1 = \max\{X_2, X_3, X_4\}$. (This shows that a statement at the top of p. 85 of Vijay Krishna's book needs an assumption such as affiliation of the X_i's.)

Solution: Since $\{X_1 = 1\}$ has conditional probability 1/4 whether heads or tails shows, the conditional probability that heads shows given $\{X_1 = 1\}$ is 0.5. Thus, $E[V|X_1 = 1] = 0.5(1 + 0.9 + 0.9 + 0.9) + 0.5(1 + 1.3 + 0 + 0) = 3$. Given $X_1 = 1$ and $Y_1 < 1$, it must be that heads occurred, so $E[V|X_1 = 1, Y_1 < 1] = 1 + 0.9 + 0.9 + 0.9 = 3.7$, which is greater than $E[V|X_1 = 1]$.

6. **[An example with interdependent values]**
Let the signals $X_1, ..., X_N$ be independent, uniformly distributed on the interval $[0, 1]$, and let the values be given by $V_i = \alpha X_i + \sum_{j=1}^N X_j$, where α is a known constant in $(-1, \infty)$. (In the following you may use the fact that N independent random variables that are uniformly distributed on the interval $[0, 1]$ divide the interval into $N+1$ random subintervals, the lengths of which are identically distributed. Consequently, for example, $E[Y_1|Y_1 < X_1] = \frac{N-1}{N+1}$, because given $Y_1 < X_1$, Y_1 is the second largest of n independent uniformly distributed random variables.)

(a) Assuming a second price auction, identify the symmetric Bayes-Nash equilibrium strategy and the resulting average revenue to the seller.

Solution: For x, y fixed, given $X_1 = x$ and $Y_1 = y$, the information we know about $X_2, \ldots, X_N$ is that the largest of these variables is y. The conditional distribution of the other $N-2$ of the variables is that they are conditionally independent, and each is uniformly distributed on the interval $[0, y]$, so they have mean $y/2$. Therefore, $v(x, y) = E[V_1|X_1 = x, Y_1 = y] = (1+\alpha)x + y + \frac{(n-2)y}{2} = (1+\alpha)x + \frac{Ny}{2}$. Therefore, the strategy giving symmetric equilibrium is $\beta(x) = v(x, x) = (1 + \alpha + \frac{N}{2})x$. The corresponding revenue is given by

$$R^{II} = E[v(Y_1, Y_1)|X_1 > Y_1] = \left(1 + \alpha + \frac{n}{2}\right) E[Y_1|X_1 > Y_1] = \left(1 + \alpha + \frac{N}{2}\right) \frac{N-1}{N+1}.$$

(b) Assuming an English auction, calculate the average revenue to the seller, and compare to the revenue for the second price auction.

Solution:

$$\begin{aligned}
R^{ENG} &= E[u(Y_1, Y_1, \ldots, Y_{N-1})|X_1 > Y_1] \\
&= E[(2+\alpha)Y_1 + \sum_{j=2}^{N-1} Y_j|X_1 > Y_1] \\
&= \left(2 + \alpha + \frac{n-2}{2}\right) E[Y_1|X_1 > Y_1] \\
&= \left(1 + \alpha + \frac{n}{2}\right) \frac{N-1}{N+1}.
\end{aligned}$$

The revenue is the same for the two auctions because for $1 > x > y > 0$, the conditional distribution of $Y_2, \ldots, Y_n$ given $(X_1 = x, Y_1 = y)$ depends on y, but is the same for all x with $y < x < 1$.