University of Illinois **Spring 2013**

ECE 368BH: Problem Set 3: Problems and Solutions
Analysis of static games (continued), and dynamics involving static games

Due: Thursday, March 7 at beginning of class
Reading: Menache and Ozdaglar, Part I (pp. 63-67 introduces potential games).
Material on players based on minimizing regret, and the Blackwell approachability theorem, can be found in Cesa-Bianchi and Lugosi, *Prediction, Learning, and Games*, Chapters 2,4, and 7. Students at UIUC can access eBook–see link on course webpage

1. **[Two player Cournot game]**
Consider the two player Cournot game with actions $s_i \in [0, \infty)$ and payoff function $\pi_i(s) = (a-(s_1+s_2)-c)s_i$. Here s_i is the amount produced by player i, the cost per unit of production is c, and the price for total quantity $s_1 + s_2$ is $a - s_1 - s_2$. Assume $0 < c < a$. Show that the game is solvable by iterated elimination of strongly dominated strategies. (Infinitely many iterations are required. To get some intuition, sketch $\pi_1(s_1, s_2)$ as a function of s_1, for various values of s_2.)

   Solution: For player 1, any value of s_1 with $s_1 > \frac{a-c}{2}$ is strictly dominated by $\frac{a-c}{2}$, because $\frac{\partial \pi_1(s)}{\partial s_1} = a-2s_1-s_2-c < 0$ for $s_1 > \frac{a-c}{2}$. The same holds for player 2. So after one iteration of iterated elimination of strongly dominated strategies (IESDS) the strategy spaces are reduced to $[0, \frac{a-c}{2}]$.

   For player 1, any value of s_1 with $s_1 < \frac{a-c}{4}$ is strictly dominated by $\frac{a-c}{4}$, because $\frac{\partial \pi_1(s)}{\partial s_1} = a - 2s_1 - s_2 - c > 0$ for $s_1 < \frac{a-c}{4}$ (and $s_2 < \frac{a-c}{2}$). The same holds for player 2. So after two iterations of IESDS, the strategy spaces are reduced to $[\frac{a-c}{4}, \frac{a-c}{2}]$.

   By induction, the sequence x_n defined by $x_0 = \frac{a-c}{2}$ and $x_{n+1} = \frac{a-c-x_n}{2}$ alternates above and below $\frac{a-c}{3}$ and after $2n$ iterations, the strategy set is reduced to the interval $[x_{2n-1}, x_{2n-2}]$. These intervals shrink to the point $\frac{a-c}{3}$, so any strategy other than $\frac{a-c}{3}$ is eventually eliminated.

2. **[Two-player, two-action, static potential games]**

   Consider the two-player, two-action, static potential game shown.

	1	2
1	a,b	c,d
2	e,f	g,h

   Under what condition on the constants a through h is there an exact potential function $(\Phi(i,j) : i,j \in \{1,2\})$ for this game? Give a potential function in case the condition holds.

   Solution: A potential function plus a constant is still a potential function, so in seeking a potential function we can assume $\Phi(1,1) = 0$. Such a potential function must satisfy:

$$\begin{aligned}
\Phi(1,1) &= 0 \\
\Phi(2,1) &= \Phi(1,1) + u_1(2,1) - u_1(1,1) = e - a \\
\Phi(2,2) &= \Phi(2,1) + u_2(2,2) - u_2(2,1) = e - a + h - f \\
\Phi(1,2) &= \Phi(1,1) + u_2(1,2) - u_2(1,1) = d - b.
\end{aligned}$$

   The function Φ defined by these values is a potential function if one more condition is satisfied, namely, $\Phi(2,2) = \Phi(1,2) + u_1(2,2) - u_1(1,2) = d - b + g - c$. Thus, a potential function exists if and only if $e - a + h - f = d - b + g - c$. If this equality holds, the equations above define a potential function. If $e - a + h - f \neq d - b + g - c$, the game does not have a potential function.

3. **[A simple graphical congestion game.]**
Let G be an undirected graph, $G = (V, E)$, where V is the set of vertices and E is the set of edges. Suppose the graph represents a campground, with V denoting the players, who are campers, and an edge $e = [i, j]$ connecting two players if their campsites are neighboring. Each player $i \in V$ has a decision variable $s_i \in S_i = \{0, 1\}$, where $s_i = 1$ means the player will play music and s_i means the player will not play music. The value for playing music to player i is v_i, but if a player plays music she also suffers a congestion cost equal to the number of neighboring campers (not including i) who play music. Thus, the payoff for player i is given by $\pi_i(s) = s_i \left(v_i - \sum_{j \in \mathcal{N}(i)} s_j \right)$, where $\mathcal{N}(i)$ is the set of neighbors of i, not including i.

(a) Show that the congestion game $(V, (S_i, i \in V), (\pi_i : i \in V))$ has an exact potential function Φ, and give a simple expression for it.

Solution: We set $\Phi(0, \ldots, 0) = 0$. For any i and s_{-i}, $\Phi(1, s_{-i}) - \Phi(0, s_{-i}) = v_i - \sum_{j \in \mathcal{N}(i)} s_j$. In words, it mean that given s_{-i}, if s_i is changed from 0 to 1, the increase in Φ is v_i minus the number of edges that connect i to neighbors that are playing to music. So to find $\Phi(s)$, consider a sequence of states beginning from state $(0, \ldots, 0)$ obtained by increasing one coordinate from zero to one at a time. Then $\Phi(s)$ should be the sum of the increases in Φ, or

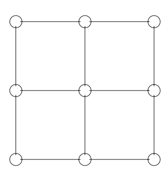
$$\Phi(s) = \sum_{i \in V} s_i v_i - \sum_{e=[i,j] \in E} s_i s_j.$$

The second sum in this expression for Φ is just the number of edges in E that connect two players both playing music. To check that Φ is a potential function for the game, note that for any i_o and any choice of s_{-i_o}.

$$\begin{aligned} \Phi(1, s_{-i_o}) - \Phi(0, s_{-i_o}) &= v_{i_o} - \sum_{e=[i,j] \in E : i = i_o} s_j \\ &= \pi(1, s_{-i_o}) - \pi(0, s_{-i_o}), \end{aligned}$$

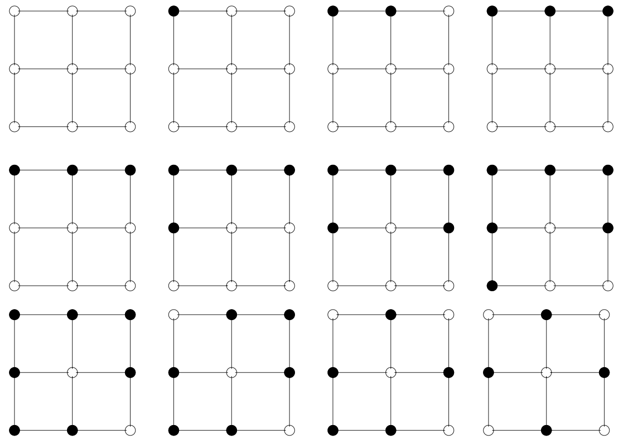
as required.

(b) Illustrate the convergence of best response dynamics for the graph shown, assuming $v_i = 1.5$ for all i.



Begin with the all zero strategy vector, and cycle through the players one at a time, first visiting players in the top row left to right, then the second row, and so on.

Solution: Just after a player i is visited, s_i will equal one if at most one of its neighbors is in state one. The sequence of strategy vectors is plotted below, were a blackened node i represents $s_i = 1$.



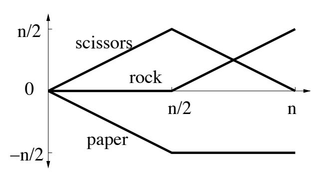
4. **[Exponential regret strategy vs. an oblivious player for rock, paper, scissors game]**
Consider two players playing the zero sum rock (R) , paper (P), scissors (S) game, for $n = 1000$ rounds. Focus on the cumulative loss of player 1, in the case player 2 plays R for the first 500 rounds and then P for the remaining 500 rounds. (Player 2's strategy is far from being a stationary one.)

   (a) Describe how you think you would do against player 2, if all you knew to begin with is that you were to play 1000 times, and you saw the outcome after each play. (This is highly subjective–there is no single correct answer.)

   Solution: You'd soon catch on to the fact player two always plays rock, so you would start playing paper all the time. When player 2 switches to playing paper at time 501, you would soon switch to scissors and keep playing scissors. You'd probably win at least 990 out of 1000 games.

   (b) Identify the cumulative loss functions $L_{R,t}, L_{P,t}, L_{S,t}$ for constant play $R, P,$ or S by player one. (Note: For example, $L_{P,t} = -t$ for $0 \leq t \leq n/2$ because playing P wins each of the first 500 games, or equivalently, results in loss -1 in each of those games.)

   Solution: Loss functions are indicated in the plot.



   (c) Suppose player one plays the exponential weighted strategy: $\widehat{p}_{t+1}(i) = \frac{e^{-\eta L_{i,t}}}{D_t}$ for $i \in \{R, P, S\}$, where $\eta = 0.1$ (which is near $\sqrt{8(\ln 3)/1000}$), and D_t is the normalizing constant making $\widehat{p}_{t+1}$ a probability distribution. Show that the expected number of times player one wins is less than or equal to 501.

   Solution: The mean number of wins for player 1 in the first 500 rounds is less than 500. It suffices to show that the total expected number of wins in the last 500 rounds is less than one.

   For $501 \leq t \leq 1000$, $L_{S,t} \geq 0$, $L_{P,t} = -500$, and $D_t \geq e^{\eta(500)} = e^{50} \geq 10^{21}$. Therefore, for such t, $\widehat{p}_{t+1}(S) = \frac{e^{-\eta L_{S,t}}}{D_t} \leq \frac{1}{10^{21}} \leq 10^{-21}$. The expected number of times the player wins during $501 \leq t \leq 1000$, is thus upper bounded by $(500)10^{-21} \leq 1$.

5. **[Predictor based on experts derived using a quadratic potential function]**
Suppose the loss function $l(p, y)$ is a convex function of p with values in $[0, 1]$. A predictor $(\widehat{p}_t : 1 \leq t \leq n)$ is to be used such that $\widehat{p}_t$ is a weighted average of the predictions $(f_{i,t} : 1 \leq i \leq M)$ produced by M experts. The cumulative loss functions for the forecaster and experts are given by $\widehat{L}_t = \sum_{1 \leq s \leq t}^{t} l(\widehat{p}_s, y_s)$ and $L_{i,t} = \sum_{1 \leq s \leq t} l(f_{i,s}, y_s)$, respectively. The (cumulative) regret vector at time t is defined by $R_t = (\widehat{L}_t - L_{i,t} : 1 \leq i \leq M)$. Use the potential function $\Phi(x) = \sum_{i=1}^{N}((x_i)_+)^2$.

   (a) Derive the expression for $\widehat{p}_t$ as a function of $(y_s, \widehat{p}_s, (f_{i,s} : 1 \leq i \leq M) : 1 \leq s \leq t-1)$ using the weight vector $w_{t-1} = \nabla\Phi(R_{t-1})$. (If all the weights are zero, let $\widehat{p}_t$ be an arbitrary element of the prediction space.)
   Solution: The weights are given by $w_{i,t-1} = \frac{\partial \Phi}{\partial x_i}(R_{t-1}) = 2(R_{i,t-1})_+$. The predictor is therefore:
   $$\widehat{p}_t = \frac{\sum_{i=1}^{N}(R_{i,t-1})_+ f_{i,t}}{\sum_{i=1}^{N}(R_{i,t-1})_+},$$
   with the understanding that $\widehat{p}_t$ can be chosen to be an arbitrary prediction if $R_{i,t-1} \leq 0$ for all i, or equivalently, if $\Phi(R_{i,t-1}) = 0$.

   (b) Show that $\Phi(R_t) \leq \Phi(R_{t-1}) + M$ for all $t \geq 0$. (Hint: Treat the case $\Phi(R_{t-1}) = 0$ separately. You may also use the fact that if g is a continuously differentiable function on the line with a piecewise continuous second derivative, then
   $g(b) \leq g(a) + g'(a)(b-a) + \sup_{\eta \in [a,b]} \frac{g''(\eta)(a-b)^2}{2}$.)
   Solution: If $\Phi(R_{t-1}) = 0$ then $R_{i,t-1} \leq 0$ for all i, so $R_{i,t} = R_{i,t-1} + l(\widehat{p}, y_t) - l(f_{i,t}, y_t) \leq l(\widehat{p}_t, y_y) \leq 1$. Summing over i with $1 \leq i \leq N$ yields $\Phi(R_t) \leq N = \Phi(R_{t-1}) + N$. It remains to consider the case $\Phi(R_{t-1}) > 0$.

   Note that Φ is continuously differentiable, and the second derivative of Φ is piecewise continuous along any line in $\mathbb{R}^N$. The, by the hint,
   $$\Phi(R_t) \leq \Phi(R_{t-1}) + \nabla\Phi(R_{t-1}) \cdot r_t + \max_{\xi} \frac{1}{2} r_t^T H(\Phi)\Big|_{\xi} r_t,$$
   where the maximum is over ξ on the line segment connecting R_{t-1} and R_t. For any $\xi \in \mathbb{R}^N$,
   $$\frac{1}{2} r_t^T H(\Phi)\Big|_{\xi} r_t = r_t^T \text{diag}(I_{\{\xi_i \geq 0\}}) r_t \leq \sum_{i=1}^{M} r_{i,t}^2 \leq M$$

   (c) Show that $\max_i R_{i,n} \leq \sqrt{nM}$ for $n \geq 1$.
   Solution: Since $\Phi(R_0) = \Phi(0) = 0$, part (b) yields
   $((\max_i R_{i,n})_+)^2 \leq \Phi(R_n) \leq \sum_{t=1}^{n}(\Phi(R_t) - \Phi(R_{t-1})) \leq nM$.

6. **[Simulation of guessing within one]**
For this problem you are to write another computer simulation for the game considered in problem 5 of problem set 2. This time you are to simulate the case that both players use the exponentially weighted forecaster with time-varying parameter $\eta_t = \sqrt{8(\ln(6)/t}$ for generation of plays at time $t+1$ for all $t \geq 1$. This value is suggested by Corollary 4.3 in Cesa-Bianchi and Lugosi. (Note that if instead we used $\eta_t = \frac{1}{\gamma t}$ we would be using exactly the soft max

rule for fictitious play.) To be definite, suppose both players would like to maximize their payoffs, where strategy sets are $\{1, 2, 3, 4, 5, 6\}$ and the payoff matrices are

$$A_1 = \begin{pmatrix} 110000 \\ 111000 \\ 011100 \\ 001110 \\ 000111 \\ 000011 \end{pmatrix}$$

and $A_2 = ones(6, 6) - A_1$. In this context, the instantaneous regret of a player for strategy i at time t is the payoff for strategy i against the play of the other player at time t, minus the payoff of the player for time t. As we know from problem set 2, the maxmin probability of winning for either player for this game is 0.5. Thus, if player one uses a mixed strategy p, the gap from optimality of p is $gap_1(p) = 0.5 - \min\{pA_1\}$. Similarly, the gap from optimality of a strategy q for player two is $gap_2(q) = 0.5 - \min\{qA_2\}$. At each time t you will need to first compute the mixed strategies for the two players, and then generate the actually (pseudo) random plays. The following matlab function could be useful.

```
function x=distRand(p)
% Generates a random variable in {1, ... , length(p)} using probability vector p
k=length(p);
p=reshape(p,k,1);
x=sum(repmat(rand(1,1),k,1) > repmat(cumsum(p)/sum(p),1,1),1)+1;
end
```

Here is what you are to turn in: (a) a copy of your computer code, (b) for $n = 100$, give

- the 6×6 matrix with i, j^{th} entry equal to the number of times player one played i and player 2 played j over times $1 \le t \le n$,
- the empirical distribution of plays for each player and the gap from optimality for those distributions, at time n.

and (c) same as (b) for $n = 1000$.

Solution:

```
function  expert_play
%  Numerical exploration of play based on weighted prediction
%  based on regret for the exponential rule.    -BH 2/18/13

A1= [1 1 0 0 0 0   % payoff function for player one
     1 1 1 0 0 0
     0 1 1 1 0 0
     0 0 1 1 1 0
     0 0 0 1 1 1
     0 0 0 0 1 1];
A2=ones(6,6)-A1;   % payoff function for player two

hist=zeros(6,6);  % stores the history of plays
R=zeros(2,6);    %rows are the cumulative regret vectors
for t=1:1000
    eta=sqrt(8*log(6)/t);  % For Hannan consistency
    p=exp_rule(R(1,:),eta); % distn used by player one at time t
    q=exp_rule(R(2,:),eta); % distn used by player two at time t
    I=distRand(p);  % I, J are the actual plays at time t
    J=distRand(q);
    hist(I,J)=hist(I,J)+1;
    R(1,:)= R(1,:) + A1(:,J)' - ones(1,6)*A1(I,J);
    R(2,:)= R(2,:) + A2(:,I)' - ones(1,6)*A2(J,I);
end  %Next, display results
n=t
hist
p_empirical=(ones(1,6)*hist')/sum(ones(1,6)*hist')
q_empirical=(ones(1,6)*hist)/sum(ones(1,6)*hist)
gaps= [0.5 - min(p_empirical*A1), 0.5 - min(q_empirical*A2)]

function X_out = exp_rule(X,gamma)
  X=exp(X*gamma);
  X_out=X/sum(X);
end

function x=distRand(p)
% Generates a random variable in {1, ... , length(p)}
% with probability vector p
  k=length(p);
  p=reshape(p,k,1);
  x=sum(repmat(rand(1,1),k,1) > repmat(cumsum(p)/sum(p),1,1),1)+1;
end

end
```

```
n =100

hist =
   2   0   2   1   0   0
   7   5   5  12   1  10
   0   0   1   0   1   0
   2   0   0   0   0   1
  12   4   0  17   1   7
   2   0   0   1   0   6

p_empirical = 0.0500   0.4000   0.0200   0.0300   0.4100   0.0900
q_empirical = 0.2500   0.0900   0.0800   0.3100   0.0300   0.2400
gaps = 0.0500   0.0800

n = 1000
hist =
    2    0    0    0    0    2
  104   56   79   59   52  123
    0    1    1    0    0    0
    1    0    1    1    0    0
  101   77   89   75   53  120
    0    0    1    0    1    1

p_empirical = 0.0040   0.4730   0.0020   0.0030   0.5150   0.0030
q_empirical = 0.2080   0.1340   0.1710   0.1350   0.1060   0.2460
gaps = 0.0230   0.0130
```

7. **[Approachability for a simple additive game]**
Consider the following two player game with strategy spaces $S_1 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ and

$S_2 = \{y \in \mathbb{R}^2 : ||y|| \leq r\}$, where r is a positive constant and $||\ ||$ is the usual Euclidean norm. The payoff for player one is the vector $x + y$. Thus, the payoff is a vector, rather than a single number, as considered by Blackwell. You may assume Blackwell's approachability theorem applies to this problem. (A simple modification of the proof for the case of finite games can be used to prove it.)

(a) An arbitrary half space H in the plane can be expressed as $H = \{x : x \cdot u \leq c\}$, where u is a unit length vector in $\mathbb{R}^2$ and $c \in \mathbb{R}$. Under what condition on u and c is this half space approachable?

Solution: The half space H is approachable if and only if the set $(-\infty, c]$ is approachable for the auxiliary game with payoff $x \cdot u + y \cdot u$. Note that player 1 can make $x \cdot u$ assume any value in $[-1, 1]$ and player 2 can make $y \cdot u$ assume any value in $[-r, r]$. Player 1 tries to minimize the payoffs, and has the dominate strategy $x = -u$ and player 2 has the dominant strategy $y = ur$. The value of the auxiliary game is thus r-1. Therefore, H is approachable if and only if $c \geq r - 1$.

(b) Assuming that $r \leq 1$, find a simple necessary and sufficient condition for a closed set $A \subset \mathbb{R}^2$ to be approachable (for player one). Your answer should be simpler than just that all half spaces containing A should be approachable. (Hint: For what p is the singleton set, $\{p\}$, approachable. Also, think about what is approachable for player 2.)

Solution: Claim: Assume $r \leq 1$. A closed set A is approachable if and only if $A \cap D_{1-r} \neq \emptyset$, where $D_{1-r} = \{x : ||x|| \leq 1 - r.\}$. .
(if part) By part (a), a half plane H is approachable if $H \cap D_{1-r} \neq \emptyset$. Thus, if $p \in D_{1-r}$ then $\{p\}$ is a convex set and all half spaces containing it are approachable. So by Blackwell's theorem, $\{p\}$ is approachable. That is, every set consisting of one point of D_{1-r} is approachable. Any superset of an approachable set is approachable. So A is approachable if $A \cap D_{1-r} \neq \emptyset$.
(only if) We prove the contrapositive. Suppose $A \cap D_{1-r} = \emptyset$. Since both A and D_{1-r} are closed, and D_{1-r} is bounded, the minimum distance between A and D_{1-r} is strictly positive. By the "if" part of the proof in (c) and a rescaling, the set D_{1-r} is approachable for player two. Therefore, A is not approachable for player one.

(c) Assuming that $r > 1$, find a simple necessary and sufficient condition for a set $A \subset \mathbb{R}^2$ to be approachable (for player one). Your answer should be simpler than just that all half spaces containing A should be approachable.

Solution: Claim: Assume $r > 1$. A closed set A is approachable if and only if $D_{r-1} \subset A$.
(if part) Any half space containing D_{r-1} is approachable by part (a), and the set D_{r-1} is convex, so it is approachable by Blackwell's theorem. Any superset of an approachable set is approachable. So A is approachable if $D_{r-1} \subset A$.
(only if) It suffices to prove the contrapositive. Suppose $D_{r-1} \not\subset A$. Then there is some $p \in D_{r-1}$ with $p \notin A$. Since A is a closed set, the distance from p to A is strictly positive. The single point p is approachable for player two, by the "if" part of part (b). So A is not approachable for player one.