University of Illinois **Spring 2013**

ECE 368BH: Problem Set 1: Problems and Solutions
Analysis of static games

Due: Thursday, January 31 at beginning of class
Reading: Menache and Ozdaglar, Part I

1. **[Guessing 2/3 of the average]**
Consider the following game for n players. Each of the players selects a number from the set $\{1,\ldots,100\}$, and a cash prize is split evenly among the players who's numbers are closest to two-thirds the average of the n numbers chosen.

   (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ_i' that always does at least as well as μ_i, and is strictly better than μ_i for some vector of strategies of the other players.)

   Solution: Any choice of number in the interval $\{68,\ldots,100\}$ is weakly dominated, because replacing a choice in that interval by the choice 67 (here 67 is $(2/3)100$ rounded to the nearest integer) would not cause a winning player to lose, while, for some choices of the other players, it could cause a losing player to win. Thus, after one step of elimination, we assume all players select numbers in the interval $\{1,\ldots,67\}$. After two steps of elimination we assume players select numbers in the set $\{1,\cdots,45\}$. After three steps, $\{1,\cdots,30\}$, and so on. At each step the set of remaining strategies has the form $\{1,\ldots,k\}$, and as long as $k\geq 2$ the set shrinks at the next step. So the procedure terminates when all players choose the number one.

   (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium.

   Solution: A bimatrix game with $A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives such an example. Playing 2 is weakly dominated for each player, and eliminating those choices leads to the Nash equilibrium $(1,1)$. However, $(2,2)$ is also a Nash equilibrium.

   (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^*=1$.)

   Solution: Consider a Nash equilibrium of mixed strategies. Let k^* be the largest integer such that there exists at least one player i choosing k^* with strictly positive probability. To complete the proof, we show that $k^*=1$, meaning all players always choose the number one. For the sake of argument by contradiction, suppose $k^*\geq 2$. Let player i denote a player that plays k^* with positive probability. For any choice of strategies of other players, player i has a pure strategy with a strictly positive probability of winning. Since k^* must be a best response for player i, it must therefore also have a strictly positive probability of winning. It is impossible for player i to win if no other chosen numbers are equal to k^*. (Indeed, if player i were the only one to choose k^*, the second highest chosen number would be strictly closer to 2/3 of the average than k^*.) Thus, at least one of the other players must have a strictly positive probability of choosing k^*. But this means that player i^* could strictly increase her payoff by selecting k^*-1 instead of k^* (indeed, such change would never change her from winning to losing, and in case she wins, she would win strictly more with positive probability) which contradicts the requirement that k^* be a best response for player i. This completes the argument by contradiction.

2. **[A game for allocation proportional to bid]**
Suppose an amount C of a divisible resource such as communication bandwidth is to be allocated to n buyers. Each buyer i submits a positive bid, b_i, and the vector of all bids is denoted by $\mathbf{b} = (b_1, \ldots, b_n)$. In return, the buyer pays the amount b_i and receives an amount $x_i = \frac{Cb_i}{B}$ of the resource, where $B = b_1 + \cdots + b_n$. The payoff for buyer i is $\pi_i(\mathbf{b}) = U_i(x_i) - b_i$, where U_i is a concave, continuously differentiable function on $(0, \infty)$ with $\lim_{x\to 0} U_i'(x) = +\infty$ and $U_i'(x) > 0$ for $0 < x_i \le C$.

   (a) Characterize the value(s) of $\mathbf{x} = (x_1, \ldots, x_n)$ such that the social welfare, $\sum_i U_i(x_i)$, is maximized, subject to the constraints $x_i \ge 0$ for $1 \le i \le n$ and $\sum_i x_i \le C$. In addition, show that if the functions U_i are strictly concave then the allocation maximizing the social welfare is unique.

   Solution: By the KKT conditions, $\frac{\partial L(\lambda,x)}{\partial x_i} = 0$, where L is the Lagrangian function defined by $L(\lambda, x) = \sum_{i=1}^n U_i(x_i) + \lambda(C - \sum_{i=1}^n x_i)$, and where $\lambda \ge 0$ is the Lagrange multiplier. This yields $U_i'(x_i) = \lambda$ for $1 \le i \le n$. Since the functions u_i are strictly increasing, it is clear that the sum constraint is binding, i.e. $\sum_{i=1}^n x_i = C$, and this equation can be used to determine $\lambda > 0$. An interpretation is that the marginal valuation for additional capacity is the same for all buyers, and is equal to λ. If the functions U_i are strictly convex, then the social welfare is a strictly concave function of $\mathbf{x}$ over a compact, convex set, and hence achieves its maximum at a unique vector $\mathbf{x}$. Alternatively, more explicitly, we could note that the functions U_i' are strictly increasing, so the x_i's are uniquely determined by λ, and the sum $\sum_i x_i$ becomes a strictly decreasing function of λ. Thus λ is uniquely determined and hence $\mathbf{x}$ is uniquely determined.

   (b) Find an explicit expression for the allocation $\mathbf{x} = (x_1, \ldots, x_n)$ that maximizes the social welfare, in case $U_i(x_i) = w_i \ln(x_i)$, where for each i, w_i is a given positive weight.

   Solution: The equations in part (a) become $\frac{w_i}{x_i} = \lambda$, or $x_i = \frac{w_i}{\lambda}$. To satisfy the capacity constraint requires $\lambda = \left(\sum_j w_j\right)/C$, yielding $x_i = \frac{Cw_i}{\sum_j w_j}$. That is, the social welfare is maximized by the allocation that is proportional to the given weights in the valuation functions.

   (c) Show that there exists a unique Nash equilibrium, and characterize it. (Hint: A necessary condition for a Nash equilibrium is $\frac{\partial \pi_i(b)}{\partial b_i} = 0$ for $1 \le i \le n$. Show that this condition is equivalent to KKT conditions for the solution of maximizing a strictly concave function, as in part (a). Define new valuation functions $\widetilde{U}_i$ by $\widetilde{U}_i'(x_i) = U'(x_i)(1 - \frac{x_i}{C})$ for $0 < x_i \le C$.)

   Solution: Let $B_{-i} = B - b_i$. Observe that $x_i = \frac{Cb_i}{B} = C\left(1 - \frac{B_{-i}}{B}\right)$. Therefore, $\frac{\partial x_i}{\partial b_i} = \frac{CB_{-i}}{B^2} = \frac{C(1-x_i)}{B}$. Therefore, by the chain rule of calculus, $\frac{\partial \pi_i(\mathbf{x})}{\partial b_i} = U'(x_i)\frac{C(1-x_i)}{B} - 1$. Therefore, the equation $\frac{\partial \pi_i(\mathbf{b})}{\partial b_i} = 0$ is equivalent to $\widetilde{U}_i'(x_i) = \frac{B}{C}$ for each i. Comparing this to the solution to part (a), it means that $\mathbf{b}$ is a Nash equilibrium if and only if $\mathbf{x}$ is a solution to the social welfare maximization problem for the alternative valuation functions $\widetilde{U}_i$ and, and the sum of the bids is λC. Since the alternative valuation functions are strictly concave, the Nash equilibrium is unique.

3. **[Nash saddle point]**
Consider a two person zero sum game represented by a finite $m \times n$ matrix A. The first player selects a probability vector p and the second selects a probability vector q. The first player wishes to minimize pAq^T and the second player wishes to maximize pAq^T. Let V denote the value of the game, so $V = \min_p \max_q pAq^T = \max_q \min_p pAq^T$.

   (a) Consider the following statement S: If $\overline{p}$ and $\overline{q}$ are probability distributions (of the appropriate dimensions) such that $\overline{p}A\overline{q}^T = V$, then $(\overline{p}, \overline{q})$ is a Nash equilibrium point (in mixed strategies). Either prove that statement S is true, or give a counter example.

   Solution: The statement if false. For example, let $m = n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\overline{p} = (1, 0)$ and $\overline{q} = (0, 1)$. Then $\overline{p}A\overline{q}^T = 0 = V$. But if the second player were to switch from using $\overline{q}$ to using $\widehat{q} = (1, 0)$, the payoff would increase from zero to $\overline{p}A\widehat{g}^T = 1$. Therefore, $(\overline{p}, \overline{q})$ is not a Nash equilibrium.

   (b) Consider the following statement T: A Nash equilibrium consisting of a pair of pure strategies exits if and only if $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Either prove that statement T is true, or give a counter example.

   Solution: The statement is true. It is the same as a statement proved in class, but with p replaced by i and q replaced by j. The proof is repeated here.

   (if part) Suppose (i^*, j^*) is a Nash equilibrium. By definition, this means $\min_i A_{i,j^*} = A_{i^*,j^*} = \max_j A_{i^*,j}$. It is easy to see that $\min_i \max_j A_{i,j}$ and $\max_j \min_i A_{i,j}$ are both contained in the length zero interval $[\min_i A_{i,j^*}, \max_j A_{i^*,j}]$, so they must be equal.

   (only if part) Suppose $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Let i^* be a minmax optimal action for the first player and j^* be a maxmin optimal action for the second player. Then $\max_j A_{i^*,j} = \min_i \max_j A_{i,j} = \max_j \min_i A_{i,j} = \min_i A_{i,j^*}$. Therefore, $A_{i^*,j^*} \leq \max_j A_{i^*,j} = \min_i A_{i,j^*}$ and $A_{i^*,j^*} \geq \min_i A_{i,j^*} = \max_j A_{i^*,j}$, which by definition means (i^*, j^*) is a Nash equilibrium.

4. **[Equilibria for a two player game]**

Consider the two player game shown:

	L	R
T	6,6	2,8
B	8,2	0,0

The first player selects T or B and the second player selects L or R.

   (a) Identify all pure strategy Nash equilibria (if any) and the payoff vector for each one.

   Solution: There are two pure Nash equilibria: (T,R) with payoff vector (2,8), and (B,L) with payoff vector (8,2).

   (b) Identify all non-degenerate mixed strategy Nash equilibria and the payoff vector for each one.

   Solution: It is easy to see there are no Nash equilibria in which exactly one of the players has a non degenerate mixed strategy. So we seek a pair of mixed strategies $(p, 1-p), (q, 1-q)$ so that either action is a best response for either player. That requires $6p + 2(1-p) = 8p$ and $6q + 2(1-q) = 8q$, or $(p, q) = (0.5, 0.5)$ which has payoff vector (4,4).

(c) Identify the polytope of all correlated equilibria by giving the set of inequalities they satisfy, and find the correlated equilibria with largest sum of payoffs.

Solution: Each correlated equilibrium corresponds to a probability distribution (a, b, c, d) over the possible pairs of actions, $\{(T, L), (T, R), (B, L), (B, R)\}$. The conditions needed to be a correlated equilibrium, in addition to (a, b, c, d) being a probability distribution, are

$$\begin{array}{lll} (T \to B) & a(6-8) + b(2-0) & \geq 0 \\ (B \to T) & c(8-6) + d(0-2) & \geq 0 \\ (L \to R) & a(6-8) + c(2-0) & \geq 0 \\ (R \to L) & b(8-6) + d(0-2) & \geq 0, \end{array}$$

where, for example, the equation for $(T \to B)$ insures that the first player would not receive a higher expected payoff by using B whenever told to play T. The equations reduce to (a, b, c, d) is a probability vector such that $a \leq b, a \leq c, d \leq b$, and $d \leq c$.

The payoff vector for a given choice of (a, b, c, d) is $(6a + 2b + 8c, 6a + 8b + 2c)$. The sum of payoffs is $12a + 10(b + c)$. To maximize the sum of payoffs, we clearly should let $d = 0$. Then $b + c = 1 - a$ and the sum of payoffs is $10 + 2a$. The largest a can be is $\frac{1}{3}$, so the correlated equilibrium with the maximum sum of payoffs is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$, and the payoff vector is $(5.333..., 5.3333...)$.