University of Illinois **Spring 2013**

ECE 586BH: Problem Set 4
Extensive form games: Normal form representation, behavioral strategies, sequential rationality

Due: Thursday, March 28 at beginning of class
Reading: Fudendberg and Triole, Sections 3.1-3.4, & 8.3
Section III of Osborne and Rubenstein, available online, covers this material too.
Lectures also relied on Myerson's book.

1. **[Ultimatum game]**
Consider the following two stage game, about how two players split a pile of 100 gold coins. The action (strategy) set of player 1 is given by $S_1 = \{0, \ldots, 100\}$, with choice i meaning that player 1 proposes to keep i of the gold coins. Player 2 learns the choice of player one, and then takes one of two actions in response: A (accept) or R (reject). If player two plays accept, the payoff vector is $(i, 100 - i)$. If player two plays reject, the payoff vector is $(0, 0)$.

   (a) Describe the extensive form version of the game using a tree labeled as in class.

   (b) Describe the normal form of the game. It suffices to specify the strategy spaces and payoff functions. (Hint: Player 2 has 2^{101} pure strategies.)

   (c) Identify a Nash equilibria of the normal form game with payoff vector $(50, 50)$.

   (d) Identify the subgame perfect equilibria of the extensive form game. (Hint: There are two of them.)

   (e) Identify the trembling hand perfect pure strategy equilibria of the normal form version of the game.

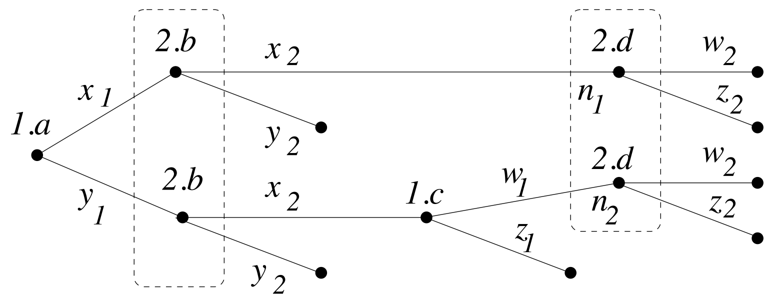
2. **[Half Kuhn poker]**
Players 1 and 2 engage in the following game. There is a deck of three cards, L, M, and H, such that L is the low card, M is the medium card, and H is the high card. After both players inspect the deck, the deck is shuffled and one card is dealt to each player, face down. The third card is put aside. Each player looks at his card. Player 1 can raise or check. If player 1 raises, player 2 can either check or fold. If player 2 checks, the players show their cards, and the player with the larger card wins two dollars from the player with the smaller card. If player 2 folds, player 2 pays one dollar to player 1. If player 1 checks, the players show their cards, and the player with the larger card wins one dollar from the player with the smaller card.

   (a) Draw a tree diagram modeling this as an extensive form game, and label it using the conventions from class (as in Myerson's book). For each node of the tree, indicate wether it is controlled by chance (use label 0), player 1, or player 2, and indicated an information state for each node controlled by a player. Group together nodes with the same player and information state using dashed lines.

   (b) Derive the normal form of this game. (Use of a computer or spreadsheet is recommended, even if you don't completely automate the calculation.)

(c) Identify a pair of saddle point strategies and the value of the game to player 1. (May need to do this numerically. The program saddlepoint.m posted to the webpage could be useful.)

3. **[Finding a behavioral representation]**
Consider the extensive form game shown, and consider the mixed strategy profile τ for the



normal representation of the game, given by $\tau = (\tau_1, \tau_2)$ where

$$\begin{aligned}\tau_1 &= .5[x_1 w_1] + .5\alpha[y_1 w_1] + .5(1-\alpha)[y_1 z_1] \\ \tau_2 &= \beta[x_2 w_2] + (1-\beta)[y_2 z_2]\end{aligned}$$

for some constants $\alpha, \beta \in (0,1)$.

(a) Does this game satisfy the perfect recall condition? Briefly explain.

(b) Find the behavioral representation σ of τ. (Hint: You should specify distributions player 1 uses in states a and c, and the distribution player 2 uses in states b and d.)

(c) Note that n_1 and n_2 are the two nodes labeled with information state d. Find the conditional probability node n_1 is reached given information state d is reached. Does it depend on τ_1 (i.e. on α)? Does σ_2 depend on τ_1 (i.e. on α)?

4. **[Joint funding with incomplete information]**
Two players each observe a random bit. Let X_i be the random bit observed by player i. Assume X_1 and X_2 are independent, Bernoulli(0.5) random variables. Each player i determines a bit D_i. Let $R > 0$. The payoff of player i is given by $u_i(X, D) = R\mathbf{1}_{\{X_1+X_2+D_1+D_2\geq 3\}} - D_i$. That is, both players receive benefit R if the sum of the random bits and the decision bits is greater than or equal to three, and player i must pay D_i, even if the sum $X_1 + X_2 + D_1 + D_2$ is not greater than or equal to three. Suppose neither player observes the random bit of the other player, and suppose player 2 observes the value of D_1 before making her decision.

(a) Sketch a decision tree giving the extensive form of this game. To be definite, suppose that the root node is a chance node with four branches, one for each of the four possible values of (X_1, X_2). Also, it is clear that player 2 should always play 0 if she observes $X_2 = D_1 = 0$, and the payoff vector in that case is $(0, 0)$. For simplicity, reduce the size of the tree by implicitly assuming player 2 aways plays 0 if she observes $X_2 = D_1 = 0$. This reduces the number of information states for player 2 from four to three.

(b) Indicate the payoff matrix for the normal representation of the game. For simplicity, it might be easier to give four times the payoff matrix rather than the payoff matrix itself.

(c) Identify all the NE in mixed strategies in case $0 < R < 1$, and identify which of those is a (trembling hand) perfect equilibrium of the normal form game and which of those is a sequential equilibrium scenario. (Hint: Player 1 has a strictly dominant strategy.) (As described in Fudenberg and Tirole Section 8.3, Kreps and Wilson (1982) defined the notion of a pair (σ, μ) being a *sequential equilibrium*, where σ is a behavioral strategy and μ is a belief vector. The definition requires two conditions on (σ, μ): (S) sequential rationality–meaning the decision for each player in each information state maximizes the expected payoff of the player, and (C) consistency, which means that (σ, μ) can be obtained as the limit of some sequence (σ^n, μ^n) such that all branches at each decision node have strictly positive probability under σ^n, and μ^n is computed from σ^n by Bayes formula. A strategy profile σ is a *sequential-equilibrium scenario* if there exists μ so that (σ, μ) is a sequential equilibrium.)

(d) Find a pure strategy profile that is an NE for any $R \geq 1$, with strictly positive expected payoffs for $R > 1$. For which value(s) of $R > 1$, if any, is it (trembling hand) perfect? For which value(s) of $R > 1$, if any, is it a sequential-equilibrium scenario?

(e) Find four pure strategy profiles different from the one in part (d) that are NE for any $R \geq 2$. For each of the two profiles, answer the following. For which value(s) of $R > 2$, if any, is the profile (trembling hand) perfect? For which value(s) of $R > 2$, if any, is the profile a sequential-equilibrium scenario?