**University of Illinois** **Spring 2013**

# ECE 368BH: Problem Set 2
## Analysis of static games (continued), and dynamics involving static games

**Due:** Thursday, February 14 at beginning of class
**Reading:** Menache and Ozdaglar, Part I. Also, for more material on fictitious play, see Shamma and Arslan (2004) paper. For material on evolutionary game theory, see Easley and Kleinberg, *Networks, Crowds, and Markets: Reasoning about a highly connected world*, Chapter 7, and Shoham and Leyton-Brown, *Multiagent Systems* pp. 225-230. (Links on course webpage.)

1. **[On quasi-concavity]**
   Let $f : C \to \mathbb{R}$, where $C$ is a nonempty convex subset of $\mathbb{R}^n$ for some $n \geq 1$.

   (a) Prove that if $f$ is a concave function, then for any constant $t$, the $t$-*upper level set* of $f$, $L_f(t) \overset{\triangle}{=} \{x \in C : f(x) \geq t\}$, is a convex set. (In other words, prove that a concave function is *quasi-concave*.)

   (b) Show that $f$ is quasi-concave if and only if, for any $x, y \in C$, $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$.

   (c) Suppose $f$ is quasi-concave and $g$ is a nondecreasing function on $\mathbb{R}$. Show that the composition $g \circ f$ is also quasi-concave. (By definition, $g \circ f(x) = g(f(x))$.)

2. **[On strictly concave functions]**
   This straight forward problem is good to have in mind when interpreting the sufficient conditions for existence of pure strategy NE for continuous games. A function $f$ is *strictly concave* if for any distinct $x$ and $y$ and $\lambda \in (0,1)$, $f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$.

   (a) Let $f$ be a continuously differentiable function with domain $\mathbb{R}^n$. Show that $f$ is strictly concave if and only if for any distinct $x, y \in \mathbb{R}^n$,

   $$(\nabla f(x) - \nabla f(y)) \cdot (y - x) > 0.$$

   (Hint: Establish the result first for $n = 1$. You may use the fact that a continuously differentiable function on $\mathbb{R}$ is strictly concave if and only if $f'$ is strictly decreasing. Reduce the result for general $n$ to the case for $n = 1$ by considering functions of the form $\theta(t) = f(a + bt)$ for $a, b \in \mathbb{R}^n$ with $b \neq 0$.)

   (b) Suppose $f$ is a twice continuously differentiable function on $\mathbb{R}^n$ and suppose its Hessian matrix $H(x)$ defined by

   $$H_{ij}(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \qquad i, j \in \{1, \cdots, n\}$$

   is (strictly) negative definite for all $x \in \mathbb{R}^n$ (i.e. $b^T H(x) b < 0$ for all nonzero $b \in \mathbb{R}^n$.) Show that $f$ is strictly concave. (Same hint as in previous part applies.)

3. **[Existence and uniqueness of NE for some games with quadratic payoff functions]**
   This problem concerns an $n$ player game with strategy space $S_i = [0,1]$ for all players $i$, and space of strategy vectors $S = S_1 \times \cdots \times S_n$.

(a) Given a vector of strategies $x \in S$, let $\overline{x} = \frac{x_1+\cdots+x_n}{n}$. Consider the payoff functions $u_i(x) = cx_i(1-x_i) - \frac{1}{2}(x_i - \overline{x})^2$, where $c \geq 0$. For what values of $c \geq 0$ does there exist a pure strategy Nash equilibrium?

(b) For the payoff functions of part (a), for what values of $c$ is the pure strategy Nash equilibrium unique?

(c) Now consider the payoff functions $u_i(x) = \frac{c}{2}x_i(1-x_i)+x_i\left(\sum_{j=1}^{n} a_{i,j}x_j\right)$, where $|a_{i,j}| \leq 1$ and $a_{i,i} = 0$ for all $i, j$. Find a constant $c_0$ so that for $c \geq c_0$ there exists a pure strategy Nash equilibrium.

(d) For the payoff functions of part (c), give a value $c_1$ so that there is a unique pure strategy NE if $c > c_1$. (Hint: A sufficient condition for a symmetric matrix to be negative definite is that the diagonal elements be strictly negative, and the sum of the absolute values of the off-diagonal elements in any row be strictly smaller than the absolute value of the diagonal element in the row.)

4. **[Guessing within one]**
Let $L$ be a positive integer multiple of three, and consider the following zero sum, two player game. Let $S = \{1, \ldots L\}$. Player one selects $i \in S$ and player two selects $j \in S$. Player one wins if $|i - j| \leq 1$ and player two wins otherwise. Each player wishes to maximize her probability of winning.

(a) Show that for any Nash equilibrium (in mixed strategies), the probability player one wins is $\frac{3}{L}$.

(b) Show that the maxmin strategy for player one is unique, but the maxmin strategy for player two is not unique.

5. **[Fictitious play for guessing within one]**
Consider the game of the previous problem for $L = 6$. Player one selects a probability distribution $p$ in an effort to minimize $pAq'$ (which is the probability the first player's number *misses* the second player's number by at least two) and player two selects $q$ to maximize $pAq'$, where

$$A = \begin{pmatrix} 001111 \\ 000111 \\ 100011 \\ 110001 \\ 111000 \\ 111100 \end{pmatrix}$$

Write a computer program for this problem. Starting with some pair of pure strategies $(i, j)$, calculate the evolution of fictitious play, so that for the $n^{th}$ play, each player plays a strategy that is a best response to the empirical average of the previous $n-1$ plays of the other player.

(a) Attach a copy of the computer program you use for the simulation.

(b) Let $p_n$ and $q_n$ denote the empirical distributions for the two players after the game has been played $n$ times. Define the duality gap for a pair of strategies $(p, q)$ by $\text{gap}(p, q) = \max(p*A) - \min(A*q')$. Compute $\text{gap}(p_n, q_n)$ for your simulation after $n = 10^i$ iterations, for $1 \leq i \leq 6$. Does your computation indicate that $\text{gap}(p_n, q_n) \to 0$? If so, comment on the rate of convergence.

6. **[Evolutionarily stable strategies and states]**

Consider the following symmetric, two-player game:

| | 1 | 2 |
|---|---|---|
| 1 | 0,0 | 1,2 |
| 2 | 2,1 | 0,0 |

That is, each player selects 1 or 2. If they select different numbers, the payoffs are the numbers selected. If they select the same number, the payoffs are zero.

(a) Does either player have a (weaky or strongly) dominant strategy?

(b) Identify all the pure strategy and mixed strategy Nash equilibria.

(c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

(d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

(e) Identify the steady states of the replicator dynamics.

(f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.

7. **[Evolutionarily stable strategies and states, II]**

Consider the following symmetric, two-player game:

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0,0 | 1,2 | 1,3 |
| 2 | 2,1 | 0,0 | 2,3 |
| 3 | 3,1 | 3,2 | 0,0 |

That is, each player selects 1,2, or 3. If they select different numbers, the payoffs are the numbers selected. If they select the same number, the payoffs are zero.

(a) Identify all the pure strategy and mixed strategy Nash equilibria.

(b) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

(c) Identify the steady states of the replicator dynamics.

(d) Prove that the strategy (or one of the strategies) identified in the previous part, when used by both players in a two player game, is a trembling hand perfect equilibrium. (Use a proof based directly on the definition of trembling hand perfect equilibrium.)

8. **[Simulation of evolutionary game of doves and hawks]**

The dove hawk game is the two player symmetric static form game given by:

| | D | H |
|---|---|---|
| D | 4,4 | 1,5 |
| H | 5,1 | 0,0 |

(a) Find an evolutionarily stable strategy (ESS) and show that it is unique.

(b) For this part you need to write and run a computer simulation using a random number generator (i.e. Monte Carlo simulation). You are to simulate a population of doves and hawks in discrete time. Suppose there are initially $n_D(1)$ dove's and $n_H(1)$ hawks at the initial time, $t = 1$. Given the numbers of each type at time $t$, $(n_D(t), n_H(t))$, the numbers at time $t + 1$ are determined as follows. Two distinct birds are selected from among all $n_D(t) + n_H(t)$ birds present at time $t$, and the two birds play the above two player game (where the strategy of a bird is the type of the bird). After the game, the two birds are returned to the population. In addition, for each player, more birds of the same type as that player are added to the population as well, with the number added equal to the payoff of the player. For example, if both birds are doves, they each have payoff 4, so the two doves are returned, plus a total of eight more doves (because 8=4+4) are added to the population. Turn in (1) a copy of your computer code and (2) a graph showing the number of doves and the number of hawks versus time $t$ for $1 \leq t \leq 100$, beginning with one dove and ten hawks at time $t = 1$.