



CS 579: Computational Complexity. Lecture 1

P vs. NP, Deterministic
Hierarchy Theorem

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Today

- Define NP, state P vs. NP problem
- Search problems/decision problems
- Diagonalization
- Time hierarchy theorem (only known theorem which allows to show that certain problems are not in P)

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- <http://www.win.tue.nl/~gwoegi/P-versus-NP.htm>

Computational problems

- In a computational problem:
 - We are given an input encoded over the alphabet $\{0,1\}$.
 - We want to return as output a solution satisfying some property.
 - Computational problem is then defined by the property that the output has to satisfy given the input.
- 4 natural types of problems: decision, search, optimization, counting.

Decision problems

- In a decision problem:
 - We are given an input $x \in \{0,1\}^*$
 - We are required to return a YES/NO answer (verify whether input satisfies property)
- E.g is an undirected graph 3 colorable?
- Specify decision problems with set of inputs $L \subseteq \{0,1\}^*$ for which the answer is YES (language)

Search problems

- In a search problem:
 - We are given an input $x \in \{0,1\}^*$
 - We are required to compute some answer $y \in \{0,1\}^*$ that is in some relation to x , if such y exists
- Search problems specified with relations $R \subseteq \{0,1\}^* \times \{0,1\}^*$, where $(x,y) \in R$ iff y is an admissible answer given x
- For graph 3 coloring, we would want the coloring as output if it exists (more demanding). Formally relation $R_{3\text{COL}}$ contains pairs $(G,c) \in$ where G is 3-colorable and c is a valid 3-coloring

P and NP

- We study asymptotic complexity of problems
- Is there “feasible” algorithm for 3 coloring?
- “feasible algorithm” = runs in poly time
- P is class of decision problems solvable in poly time
- Easier to verify than come up with solution...

P and NP

- P is class of decision problems solvable in poly time.
- Search problem defined by a relation R is an NP search problem if there is a poly time algorithm that given x and y decides whether $(x,y) \in R$, and if there is a polynomial p s.t. if $(x,y) \in R$, then $|y| \leq p(|x|)$.
- Captures the fact that relation is efficiently computable and solutions (if they exist) are short.

P and NP

- Decision problem L is in NP if
 - (Definition 1) there is some NP relation R such that $x \in L$ iff there is y s.t. $(x, y) \in R$
 - (Definition 2) there is a poly time algorithm $V(.,.)$ and a polynomial p s.t. $x \in L$ iff there is a y , $|y| \leq p(x)$ s.t. $V(x, y)$ accepts.
- $NP =$ class of NP decision problems.

P and NP

- **Theorem 1.** NP is the set of decision problems that are solvable in poly time by a non-deterministic Turing machine.

P and NP

- NP as a complexity class is defined as class of decision problems: easier to develop cleaner theory, complexity of decision problems completely characterizes complexity of search problems.
- **Theorem 2.** For every NP search problem there is an NP decision problem such that if the decision problem is solvable in time $t(n)$ then the search problem is solvable in time $O(n^{O(1)} \cdot t(n^{O(1)}))$. In particular, $P=NP$ iff every NP search problem is solvable in poly time.

P and NP

- For function $t: \mathbb{N} \rightarrow \mathbb{N}$, we define
 - $DTIME(t(n))$ the set of decision problems that are solvable by a deterministic Turing machine within time $t(n)$ on inputs of length n
 - $NTIME(t(n))$ the set of decision problems that are solvable by a non-deterministic Turing machine within time $t(n)$ on inputs of length n
- $\mathbf{P} = \bigcup_k DTIME(O(n^k))$
- $\mathbf{NP} = \bigcup_k NTIME(O(n^k))$

Diagonalization

- Only known way of proving separations between complexity classes
- Similar to Cantor
- Halting Problem is undecidable

Diagonalization

- **Definition.** (Time constructible functions) A function $t:\mathbb{N}\rightarrow\mathbb{N}$ is time constructible if there is algorithm that, on input n , computes the value $t(n)$ in time $O(t(n))$
- Eg. All polynomials, all combinations of exponential, polynomial and root functions are time constructible
- Not time constructible: $t(n) = n^2$ if the number n written in binary encodes a turing machine that halts on all inputs and $t(n) = n^2 + 1$ otherwise

Time hierarchy theorem

- **Theorem (simple version).** For every time-constructible function $t(\cdot)$, there is a language L such that every algorithm that decides L must run in time $>t(n)$ on infinitely many inputs, but $L \in \text{DTIME}(t^{O(1)}(n))$.

Time hierarchy theorem

- **Theorem (alternative formulation).** For every f time-constructible function with $f(n)\log f(n) = o(g(n))$

$$\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n))$$

E and EXP

- $E = \text{DTIME}(2^{O(n)})$
- $\text{EXP} = \bigcup_k \text{DTIME}(2^{O(n^k)})$

Corollary. $P \neq E$ and $E \neq \text{EXP}$

Ladner's theorem and NP-intermediate problems

- **Theorem.** If $P \neq NP$ then there exists a language $L \in NP \setminus P$ that is not NP-complete.