Information Theory: Homework I

September 9th 2025

Problem 1 (DIFFICULT): Let H_α be an "entropy of order α" assigning a real number to every finite pmf $P = (p_1, \ldots, p_n)$. Assume the entropy satisfies the following properties:
(A1) Continuity & Symmetry. $H_\alpha(P)$ is continuous in P and invariant under permutations of the coordinates.
(A2) Expansibility. $H_\alpha(p_1, \ldots, p_n, 0) = H_\alpha(p_1, \ldots, p_n)$.
(A3) Maximality at uniform. For fixed n, $H_\alpha(P)$ is maximized by the uniform $U_n = (1/n, \ldots, 1/n)$.
(A4) Additivity for independent experiments. If $X \perp Y$, then

$$H_\alpha(X, Y) \;=\; H_\alpha(X) + H_\alpha(Y).$$

(A5) Generalized (branch) composition via a KN-mean. For any joint (X, Y),

$$H_\alpha(X, Y) \;=\; H_\alpha(Y) \;+\; H_\alpha(X \mid Y),$$

with conditional entropy given by a Kolmogorov–Nagumo mean (same generator for all finite alphabets):

$$H_\alpha(X \mid Y) \;=\; \phi^{-1}\Big(\sum_y P_Y(y)\, \phi\big(H_\alpha(X \mid Y{=}y)\big)\Big),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone (hence invertible).
(A6) Normalization. $H_\alpha(U_n) = \log n$ for all $n \geq 1$ (choice of log base fixes units).
Show that the unique parametrized function satisfying the axioms is Renyi's entropy.

Problem 2: For discrete X, Y, Z, prove that

$$H(X, Y) + H(Y, Z) - H(Y) \;=\; H(X, Y, Z) + I(X; Z \mid Y).$$

Problem 3: Prove Han's Theorem, stated below.

Let $(X_1, \ldots, X_n)$ be discrete random variables. Then

$$\frac{1}{n-1} \sum_{i=1}^{n} H\big(X_{[n]\setminus\{i\}} \,\big|\, X_i\big) \;\leq\; H(X_1, \ldots, X_n) \;\leq \frac{1}{n-1} \sum_{i=1}^{n} H\big(X_{[n]\setminus\{i\}}\big).$$

For another formulation and its proof based on submodularity, refer to Polyanskiy+Wu, page 16.

Problem 4. Prove the chain rule for mutual information (Cover+Thomas, page 24)

Let $X_1, \ldots, X_n$ and Y be discrete random variables. Then

$$I(X_1, \ldots, X_n; Y) \;=\; \sum_{i=1}^{n} I(X_i; Y \mid X_1, \ldots, X_{i-1}).$$

Then, prove the tenzorization inequality for mutual information.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be random pairs. Define $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$. Then

$$I(X; Y) \;\leq\; \sum_{i=1}^{n} I(X_i; Y_i).$$

Problem 5 – Problem 2.9 from Cover+Thomas, Metric.

Problem 6 – Problem 2.10 from Cover+Thomas, Entropy of a disjoint mixture.

Problem 7 – Problem 2.11 from Cover+Thomas, Measure of correlation.

Problem 8 – Problem 2.25 from Cover+Thomas, Venn diagrams.