Recitation #1

Problems 2.9, 2.13, and 2.14 covered by Kanad Sarkar

9) **A metric.** A function $\rho(x, y)$ is a metric if for all x, y,

- $\rho(x, y) \geq 0$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, y) = 0 \quad$ if and only if $x = y$
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

a) Show that $\rho(X, Y) = H(X|Y) + H(Y|X)$ satisfies the first, second and fourth properties above. If we say that $X = Y$ if there is a one-to-one function mapping from X to Y, then the third property is also satisfied, and $\rho(X, Y)$ is a metric.

b) Verify that $\rho(X, Y)$ can also be expressed as

$$
\begin{aligned}
\rho(X, Y) &= H(X) + H(Y) - 2I(X; Y) && (5) \\
&= H(X, Y) - I(X; Y) && (6) \\
&= 2H(X, Y) - H(X) - H(Y). && (7)
\end{aligned}
$$

replacement is lower.

9) *A metric*

a) Let

$$\rho(X,Y) = H(X|Y) + H(Y|X). \tag{21}$$

Then

- Since conditional entropy is always ≥ 0, $\rho(X,Y) \geq 0$.
- The symmetry of the definition implies that $\rho(X,Y) = \rho(Y,X)$.
- By problem 2.6, it follows that $H(Y|X)$ is 0 iff Y is a function of X and $H(X|Y)$ is 0 iff X is a function of Y. Thus $\rho(X,Y)$ is 0 iff X and Y are functions of each other - and therefore are equivalent up to a reversible transformation.
- Consider three random variables X, Y and Z. Then

$$\begin{aligned} H(X|Y) + H(Y|Z) &\geq H(X|Y,Z) + H(Y|Z) && (22)\\ &= H(X,Y|Z) && (23)\\ &= H(X|Z) + H(Y|X,Z) && (24)\\ &\geq H(X|Z), && (25) \end{aligned}$$

from which it follows that

$$\rho(X,Y) + \rho(Y,Z) \geq \rho(X,Z). \tag{26}$$

Note that the inequality is strict unless $X \to Y \to Z$ forms a Markov Chain and Y is a function of X and Z.

b) Since $H(X|Y) = H(X) - I(X;Y)$, the first equation follows. The second relation follows from the first equation and the fact that $H(X,Y) = H(X) + H(Y) - I(X;Y)$. The third follows on substituting $I(X;Y) = H(X) + H(Y) - H(X,Y)$.

13) **Inequality.** Show $\ln x \geq 1 - \frac{1}{x}$ for $x > 0$.

13) *Inequality*. Using the Remainder form of the Taylor expansion of $\ln(x)$ about $x = 1$, we have for some c between 1 and x

$$\ln(x) = \ln(1) + \left(\frac{1}{t}\right)_{t=1}(x-1) + \left(\frac{-1}{t^2}\right)_{t=c}\frac{(x-1)^2}{2} \leq x - 1$$

since the second term is always negative. Hence letting $y = 1/x$, we obtain

$$-\ln y \leq \frac{1}{y} - 1$$

or

$$\ln y \geq 1 - \frac{1}{y}$$

with equality iff $y = 1$.

14) **Entropy of a sum.** Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.

14) *Entropy of a sum.*

a) $Z = X + Y$. Hence $p(Z = z|X = x) = p(Y = z - x|X = x)$.

$$
\begin{aligned}
H(Z|X) &= \sum p(x) H(Z|X = x) \\
&= -\sum_x p(x) \sum_z p(Z = z|X = x) \log p(Z = z|X = x) \\
&= \sum_x p(x) \sum_y p(Y = z - x|X = x) \log p(Y = z - x|X = x) \\
&= \sum p(x) H(Y|X = x) \\
&= H(Y|X).
\end{aligned}
$$

If X and Y are independent, then $H(Y|X) = H(Y)$. Since $I(X; Z) \geq 0$, we have $H(Z) \geq H(Z|X) = H(Y|X) = H(Y)$. Similarly we can show that $H(Z) \geq H(X)$.

14) **Entropy of a sum.** Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.

b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

c) Under what conditions does $H(Z) = H(X) + H(Y)$?

b) Consider the following joint distribution for X and Y Let

$$X = -Y = \begin{cases} 1 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/2 \end{cases}$$

Then $H(X) = H(Y) = 1$, but $Z = 0$ with prob. 1 and hence $H(Z) = 0$.

c) We have

$$H(Z) \leq H(X,Y) \leq H(X) + H(Y)$$

because Z is a function of (X,Y) and $H(X,Y) = H(X) + H(Y|X) \leq H(X) + H(Y)$. We have equality iff (X,Y) is a function of Z and $H(Y) = H(Y|X)$, i.e., X and Y are independent.