Recitation Sept. 21, 2022

Information Theory

Problems from Cover & Thomas and Stanford EE376A Jan. 2019 HW

Problem 1: Cover 3.3 Piece of Cake

3) **Piece of cake**

A cake is sliced roughly in half, the largest piece being chosen each time, the other pieces discarded. We will assume that a random cut creates pieces of proportions:

$$P = \begin{cases} (\frac{2}{3}, \frac{1}{3}) & \text{w.p. } \frac{3}{4} \\ (\frac{2}{5}, \frac{3}{5}) & \text{w.p. } \frac{1}{4} \end{cases}$$

Thus, for example, the first cut (and choice of largest piece) may result in a piece of size $\frac{3}{5}$. Cutting and choosing from this piece might reduce it to size $(\frac{3}{5})(\frac{2}{3})$ at time 2, and so on.

How large, to first order in the exponent, is the piece of cake after n cuts?

Problem 1: Solutions

3) Let C_i be the fraction of the piece of cake that is cut at the ith cut, and let T_n be the fraction of cake left after n cuts. Then we have $T_n = C_1 C_2 \ldots C_n = \prod_{i=1}^{n} C_i$.

$$\begin{aligned} \lim \frac{1}{n} \log T_n &= \lim \frac{1}{n} \sum_{i=1}^{n} \log C_i \\ &= E[\log C_1] \\ &= \frac{3}{4} \log \frac{2}{3} + \frac{1}{4} \log \frac{3}{5}. \end{aligned}$$

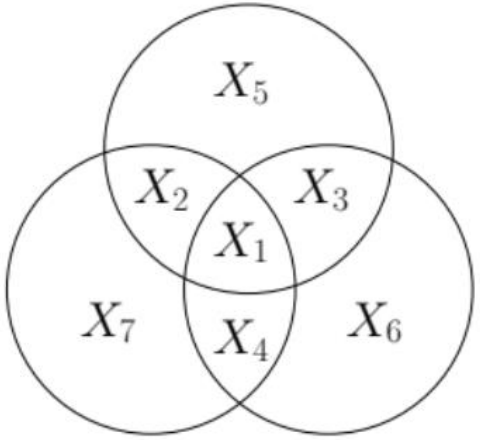
$$T_n = 2^{n\left(\frac{1}{n} \log T_n\right)}$$

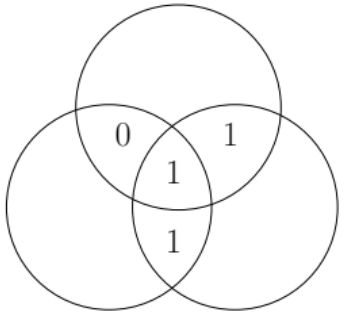
Problem 2: Hamming Code

Entropy of Hamming Code. [1]

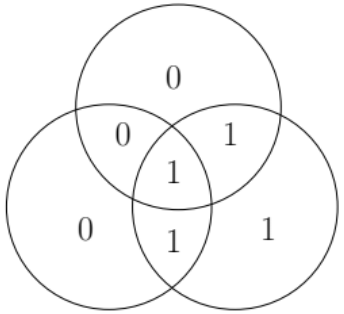
Hamming code is a simple error-correcting code that can correct up to one error in a sequence of bits. Now consider information bits $X_1, X_2, X_3, X_4 \in \{0, 1\}$ chosen uniformly at random, together with check bits X_5, X_6, X_7 chosen to make the parity of the circles even.

(eg: $X_1 + X_2 + X_4 + X_7 = 0 \mod 2$)





becomes



That is, 1011 becomes 1011010.

(a) What is the entropy of $H(X_1, X_2, ..., X_7)$?

Now we make an error (or not) in one of the bits (or none). Let $\mathbf{Y} = \mathbf{X} \oplus \mathbf{e}$, where $\mathbf{e}$ is equally likely to be $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$, or $(0, 0, \ldots, 0)$, and $\mathbf{e}$ is independent of $\mathbf{X}$.

(b) Show that one can recover the message $\mathbf{X}$ perfectly from $\mathbf{Y}$. (Please provide a justification, detailed proof not required.)

(c) What is $H(\mathbf{X}|\mathbf{Y})$?

(d) What is $I(\mathbf{X}; \mathbf{Y})$?

(e) What is the entropy of $\mathbf{Y}$?

a) X_1, X_2, X_3, X_4 bernoulli w/ $p = \frac{1}{2}$

$$X_5 = X_1 \oplus X_2 \oplus X_3$$
$$X_6 = X_1 \oplus X_3 \oplus X_4$$
$$X_7 = X_1 \oplus X_2 \oplus X_4$$

$$H(X_1, X_2, \ldots, X_7) = H(X_1, X_2, X_3, X_4) = \sum_{i=1}^{4} H(X_i) = 4$$

b) We know

$$X_5 \oplus X_1 \oplus X_2 \oplus X_3 = 0 \quad a$$
$$X_6 \oplus X_1 \oplus X_3 \oplus X_4 = 0 \quad b$$
$$X_7 \oplus X_1 \oplus X_2 \oplus X_4 = 0 \quad c$$

Error

x_i	a	b	c
1	1	1	1
2	1	0	1
3	0	1	0
4	0	1	1
5	1	0	0
6	0	1	0
7	0	0	1
NO	0	0	0

c) $H(X|Y) = 0$

d) $I(X;Y) = H(X) - H(X|Y)$

$= H(X) = 4$

e) $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(X \oplus e \mid X) = H(Y) - H(e)$

$= H(X) - H(X|Y) = H(X)$

$H(Y) = H(X) + H(e) = 4 + \underset{\overset{\shortparallel}{3}}{\log_2 8} = 7$

$Y = X \oplus e$

$e = X \oplus Y$

e : $(0,0,\dots,0)$
$(1,0,\dots,0)$
$(0,1,0,\dots,0)$
$\vdots$
$(0,0,\dots,0,1)$

8 options

uniform distribution

$H(X,Y) = H(X) + H(Y|X)$
$= H(Y) + H(X|Y)$