Recitation Sept. 21, 2022

Information Theory

Problems from Cover & Thomas and Stanford EE376A Jan. 2019 HW
Problem 1: Cover 3.3 Piece of Cake

3) Piece of cake
A cake is sliced roughly in half, the largest piece being chosen each time, the other pieces discarded. We will assume that a random cut creates pieces of proportions:

\[ P = \begin{cases} 
\left(\frac{2}{3}, \frac{1}{3}\right) & \text{w.p. } \frac{3}{4} \\
\left(\frac{1}{5}, \frac{4}{5}\right) & \text{w.p. } \frac{1}{4}
\end{cases} \]

Thus, for example, the first cut (and choice of largest piece) may result in a piece of size \( \frac{3}{5} \). Cutting and choosing from this piece might reduce it to size \( \left(\frac{3}{5}\right)\left(\frac{2}{3}\right) \) at time 2, and so on.

How large, to first order in the exponent, is the piece of cake after \( n \) cuts?
Problem 1: Solutions

3) Let $C_i$ be the fraction of the piece of cake that is cut at the $i$th cut, and let $T_n$ be the fraction of cake left after $n$ cuts. Then we have $T_n = C_1 C_2 \ldots C_n = \prod_{i=1}^{n} C_i$.

$$\lim \frac{1}{n} \log T_n = \lim \frac{1}{n} \sum_{i=1}^{n} \log C_i$$

$$= E[\log C_1]$$

$$= \frac{3}{4} \log \frac{2}{3} + \frac{1}{4} \log \frac{3}{5}.$$
Problem 2: Hamming Code

Entropy of Hamming Code. [1]
Hamming code is a simple error-correcting code that can correct up to one error in a sequence of bits. Now consider information bits $X_1, X_2, X_3, X_4 \in \{0, 1\}$ chosen uniformly at random, together with check bits $X_5, X_6, X_7$ chosen to make the parity of the circles even.
(eg: $X_1 + X_2 + X_4 + X_7 = 0 \mod 2$)
becomes

That is, 1011 becomes 1011010.
(a) What is the entropy of $H(X_1, X_2, \ldots, X_7)$?

Now we make an error (or not) in one of the bits (or none). Let $Y = X \oplus e$, where $e$ is equally likely to be $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$, or $(0, 0, \ldots, 0)$, and $e$ is independent of $X$.

(b) Show that one can recover the message $X$ perfectly from $Y$. (Please provide a justification, detailed proof not required.)

(c) What is $H(X|Y)$?

(d) What is $I(X; Y)$?

(e) What is the entropy of $Y$?
a) \( X_1, X_2, X_3, X_4 \) bernoulli \( \omega / \rho = \frac{1}{2} \)

\[
X_5 = X_1 \oplus X_2 \oplus X_3 \\
X_6 = X_1 \oplus X_3 \oplus X_4 \\
X_7 = X_1 \oplus X_2 \oplus X_4
\]

\[
H(X_1, X_2, ..., X_7) = H(X_1, X_2, X_3, X_4) = \sum_{i=1}^{4} H(X_i) = 4
\]
b) We know

\[ X_5 \oplus X_1 \oplus X_2 \oplus X_3 = 0 \]
\[ X_6 \oplus X_1 \oplus X_3 \oplus X_4 = 0 \]
\[ X_7 \oplus X_1 \oplus X_2 \oplus X_4 = 0 \]

\[ H(X_1 Y) = 0 \]

c) \[ I(X; Y) = H(X) - H(X | Y) \]
\[ = H(X) = 4 \]

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e) \[ I(x; y) = H(y) - H(y|x) = H(c_y) - H(x \oplus e | x) = H(c_y) - H(e) = H(x) - H(x|y) = H(x) \]

\[ H(c_y) = H(x) + H(e) = 4 + \log_2 8 = 7 \]

\[ y = x \oplus e \]
\[ e = x \oplus y \]

\[ e = (0,0,\ldots,0) \]
\[ (1,0,\ldots,0) \]
\[ (0,1,0,\ldots,0) \]
\[ \vdots \]
\[ (0,0,\ldots,0,1) \]

8 options

uniform distribution

\[ H(x, y) = H(x) + H(c_y | x) = H(c_y) + H(x | y) \]