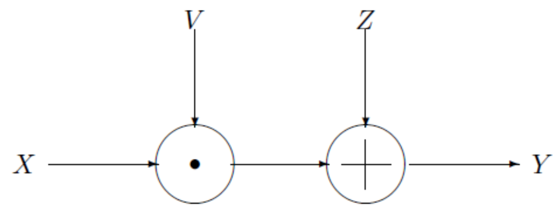
Problem Solving Session 7

Chapter 9

5) **Fading channel.**
Consider an additive noise fading channel



$$Y = XV + Z,$$

where Z is additive noise, V is a random variable representing fading, and Z and V are independent of each other and of X. Argue that knowledge of the fading factor V improves capacity by showing

$$I(X;Y|V) \geq I(X;Y).$$

5) *Fading Channel*
Expanding $I(X;Y,V)$ in two ways, we get

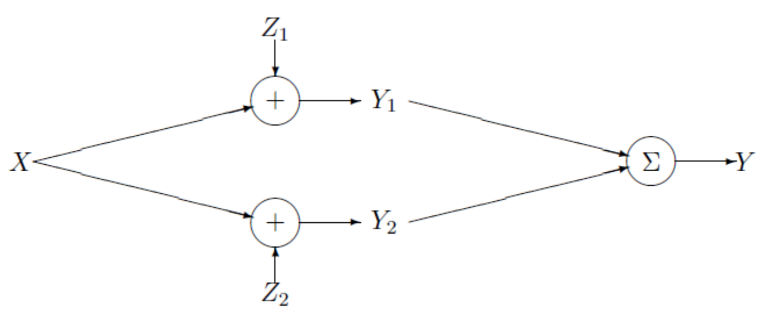
$$\begin{aligned} I(X;Y,V) &= I(X;V) + I(X;Y|V) && (829) \\ &= I(X;Y) + I(X;V|Y) && (830) \end{aligned}$$

i.e.

$$\begin{aligned} I(X;V) + I(X;Y|V) &= I(X;Y) + I(X;V|Y) \\ I(X;Y|V) &= I(X;Y) + I(X;V|Y) && (831) \\ I(X;Y|V) &\geq I(X;Y) && (832) \end{aligned}$$

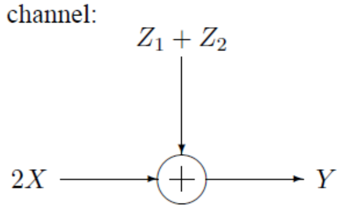
where (1773) follows from the independence of X and V, and (832) follows from $I(X;V|Y) \geq 0$.

7) **Multipath Gaussian channel.** Consider a Gaussian noise channel of power contraint P, where the signal takes two different paths and the received noisy signals are added together at the antenna.



a) Find the capacity of this channel if Z_1 and Z_2 are jointly normal with covariance matrix $K_Z = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$.
b) What is the capacity for $\rho = 0$, $\rho = 1$, $\rho = -1$?

7) *Multipath Gaussian channel.*
The channel reduces to the following channel:



The power constraint on the input $2X$ is $4P$. Z_1 and Z_2 are zero mean, and therefore so is $Z_1 + Z_2$. Then

$$\begin{aligned} Var(Z_1 + Z_2) &= E[(Z_1 + Z_2)^2] \\ &= E[Z_1^2 + Z_2^2 + 2Z_1 Z_2] \\ &= 2\sigma^2 + 2\rho\sigma^2. \end{aligned}$$

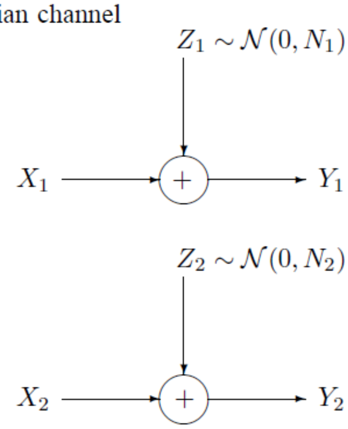
Thus the noise distribution is $\mathcal{N}(0, 2\sigma^2(1+\rho)$.

a) Plugging the noise and power values into the formula for the one-dimensional (P, N) channel capacity, $C = \frac{1}{2}\log(1 + \frac{P}{N})$, we get

$$\begin{aligned} C &= \frac{1}{2}\log\left(1 + \frac{4P}{2\sigma^2(1+\rho)}\right) \\ &= \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2(1+\rho)}\right). \end{aligned}$$

b) i) When $\rho = 0$, $C = \frac{1}{2}\log(1 + \frac{2P}{\sigma^2})$.
   ii) When $\rho = 1$, $C = \frac{1}{2}\log(1 + \frac{P}{\sigma^2})$.
   iii) When $\rho = -1$, $C = \infty$.

8) **Parallel Gaussian channels**
Consider the following parallel Gaussian channel



where $Z_1 \sim \mathcal{N}(0,N_1)$ and $Z_2 \sim \mathcal{N}(0,N_2)$ are independent Gaussian random variables and $Y_i = X_i + Z_i$. We wish to allocate power to the two parallel channels. Let β_1 and β_2 be fixed. Consider a total cost constraint $\beta_1 P_1 + \beta_2 P_2 \leq \beta$, where P_i is the power allocated to the i^{th} channel and β_i is the cost per unit power in that channel. Thus $P_1 \geq 0$ and $P_2 \geq 0$ can be chosen subject to the cost constraint β.

a) For what value of β does the channel stop acting like a single channel and start acting like a pair of channels?
b) Evaluate the capacity and find P_1, P_2 that achieve capacity for $\beta_1 = 1, \beta_2 = 2, N_1 = 3, N_2 = 2$ and $\beta = 10$.

8) *Parallel channels*
When we have cost constraints on the power, we need to optimize the total capacity of the two parallel channels

$$C = \frac{1}{2}\log\left(1+\frac{P_1}{N_1}\right) + \frac{1}{2}\log\left(1+\frac{P_2}{N_2}\right) \tag{833}$$

subject to the constraint that

$$\beta_1 P_1 + \beta_2 P_2 \le \beta \tag{834}$$

Using the methods of Section 9.4, we set

$$J(P_1, P_2) = \sum \frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right) + \lambda\left(\sum \beta_i P_i\right) \tag{835}$$

and differentiating with respect to P_i, we have

$$\frac{1}{2}\frac{1}{P_i + N_i} + \lambda\beta_i = 0, \tag{836}$$

or

$$P_i = \left(\frac{\nu}{\beta_i} - N_i\right)^+. \tag{837}$$

or

$$\beta_i P_i = (\nu - \beta_i N_i)^+. \tag{838}$$

a) It follows that we will put all the signal power into the channel with less weighted noise ($\beta_i N_i$) until the total weighted power of noise + signal in that channel equals the weighted noise power in the other channel. After that, we will split any additional power between the two channels according to their weights. Thus the combined channel begins to behave like a pair of parallel channels when the signal power is equal to the difference of the two weighted noise powers, i.e., when $\beta_1\beta = \beta_2 N - 2 - \beta_1 N_1$.
b) In this case, $\beta_1 N_1 < \beta_2 N_2$, so we would put power into channel 1 until $\beta = 1$. After that we would put power according to their weights, i.e. we would divide remaining power of 9 in the ratio 2 is to 1. Thus we would set $P_1 = 6 + 1$ and $P_2 = 3$, and so that $\nu = 10$ in the equation above. The capacity in this case is

$$C = \frac{1}{2}\log(1 + 7/3) + \frac{1}{2}\log(1 + 3/2) = 1.53 \text{ bits}. \tag{839}$$

Chapter 10

7) **Erasure distortion.** Consider $X \sim \text{Bernoulli}(\frac{1}{2})$, and let the distortion measure be given by the matrix

$$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}. \tag{921}$$

Calculate the rate distortion function for this source. Can you suggest a simple scheme to achieve any value of the rate distortion function for this source?

7) *Erasure distortion.* Consider $X \sim \text{Bernoulli}(\frac{1}{2})$, and the distortion measure

$$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}. \tag{1008}$$

The infinite distortion constrains $p(0,1) = p(1,0) = 0$. Hence by symmetry the joint distribution of $(X, \widehat{X})$ is of the form shown in Figure 11.
For this joint distribution, it is easy to calculate the distortion $D = \alpha$ and that $I(X; \widehat{X}) = H(X) - H(X|\widehat{X}) = 1 - \alpha$. Hence we have $R(D) = 1 - D$ for $0 \le D \le 1$. For $D > 1$, $R(D) = 0$.
It is very see how we could achieve this rate distortion function. If D is rational, say k/n, then we send only the first $n - k$ of any block of n bits. We reproduce these bits exactly and reproduce the remaining bits as erasures. Hence we can send information at rate $1 - D$ and achieve a distortion D. If D is irrational, we can get arbitrarily close to D by using longer and longer block lengths.