# Problem Solving Session 6

**7.28**

28) **Choice of channels.**

Find the capacity $C$ of the union of 2 channels $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 or over channel 2 but not both. Assume the output alphabets are distinct and do not intersect.

a) Show $2^C = 2^{C_1} + 2^{C_2}$. Thus $2^C$ is the effective alphabet size of a channel with capacity $C$.

28) *Choice of Channels*

(a) This is solved by using the very same trick that was used to solve problem 2.10.
Consider the following communication scheme:

$$X = \begin{cases} X_1 & \text{Probability } \alpha \\ X_2 & \text{Probability } (1-\alpha) \end{cases}$$

Let

$$\theta(X) = \begin{cases} 1 & X = X_1 \\ 2 & X = X_2 \end{cases}$$

Since the output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are disjoint, $\theta$ is a function of $Y$ as well, i.e. $X \to Y \to \theta$.

$$\begin{aligned} I(X; Y, \theta) &= I(X;\theta) + I(X;Y|\theta) \\ &= I(X;Y) + I(X;\theta|Y) \end{aligned}$$

Since $X \to Y \to \theta$, $I(X;\theta|Y) = 0$. Therefore,

$$\begin{aligned} I(X;Y) &= I(X;\theta) + I(X;Y|\theta) \\ &= H(\theta) - H(\theta|X) + \alpha I(X_1;Y_1) + (1-\alpha) I(X_2;Y_2) \\ &= H(\alpha) + \alpha I(X_1;Y_1) + (1-\alpha) I(X_2;Y_2) \end{aligned}$$

Thus, it follows that

$$C = \sup_\alpha \{H(\alpha) + \alpha C_1 + (1-\alpha) C_2\}.$$

Maximizing over $\alpha$ one gets the desired result. The maximum occurs for $H'(\alpha) + C_1 - C_2 = 0$, or $\alpha = 2^{C_1}/(2^{C_1} + 2^{C_2})$.

**7.33**

33) **BSC with feedback.** Suppose that feedback is used on a binary symmetric channel with parameter $p$. Each time a $Y$ is received, it becomes the next transmission. Thus $X_1$ is Bern(1/2), $X_2 = Y_1$, $X_3 = Y_2$, $\ldots$, $X_n = Y_{n-1}$.

a) Find $\lim_{n\to\infty} \frac{1}{n} I(X^n; Y^n)$.

b) Show that for some values of $p$, this can be higher than capacity.

33) **BSC with feedback solution.**

a)

$$I(X^n; Y^n) = H(Y^n) - H(Y^n|X^n).$$

$$H(Y^n|X^n) = \sum_i H(Y_i|Y^{i-1}, X^n) = H(Y_1|X_1) + \sum_i H(Y_i|Y^n) = H(p) + 0.$$

$$H(Y^n) = \sum_i H(Y_i|Y^{i-1}) = H(Y_1) + \sum_i H(Y_i|X_i) = 1 + (n-1)H(p)$$

So,

$$I(X^n;Y^n) = 1 + (n-1)H(p) - H(p) = 1 + (n-2)H(p)$$

and,

$$\lim_{n\to\infty} \frac{1}{n} I(X^n;Y^n) = \lim_{n\to\infty} \frac{1+(n-2)H(p)}{n} = H(p)$$

b) For the BSC $C = 1 - H(p)$. For $p = 0.5$, $C = 0$, while $\lim_{n\to\infty} \frac{1}{n} I(X^n;Y^n) = H(0.5) = 1$.

**7.35**

35) **Capacity.**
Suppose channel $\mathcal{P}$ has capacity $C$, where $\mathcal{P}$ is an $m \times n$ channel matrix.

a) What is the capacity of

$$\tilde{\mathcal{P}} = \begin{bmatrix} \mathcal{P} & 0 \\ 0 & 1 \end{bmatrix}$$

b) What about the capacity of

$$\hat{\mathcal{P}} = \begin{bmatrix} \mathcal{P} & 0 \\ 0 & I_k \end{bmatrix}$$

where $I_k$ if the $k \times k$ identity matrix.

35) **Solution: Capacity.**

a) By adding the extra column and row to the transition matrix, we have two channels in parallel. You can transmit on either channel. From problem 7.28, it follows that

$$\begin{aligned} \tilde{C} &= \log(2^0 + 2^C) \\ \tilde{C} &= \log(1 + 2^C) \end{aligned}$$

b) This part is also an application of the conclusion problem 7.28. Here the capacity of the added channel is $\log k$.

$$\begin{aligned} \hat{C} &= \log(2^{\log k} + 2^C) \\ \hat{C} &= \log(k + 2^C) \end{aligned}$$

**8.10**

10) **The Shape of the Typical Set**
Let $X_i$ be i.i.d. $\sim f(x)$, where

$$f(x) = ce^{-x^4}.$$

Let $h = -\int f \ln f$. Describe the shape (or form) or the typical set $A_\epsilon^{(n)} = \{x^n \in \mathcal{R}^n : f(x^n) \in 2^{-n(h\pm\epsilon)}\}$.

10) *The Shape of the Typical Set*

We are interested in the set $\{x^n \in \mathcal{R} : f(x^n) \in 2^{-n(h\pm\epsilon)}\}$. This is:

$$2^{-n(h-\epsilon)} \leq f(x^n) \leq 2^{-n(h+\epsilon)}$$

Since $X_i$ are i.i.d.,

$$f(x^n) = \prod_{i=1}^{n} f(x) \quad (734)$$

$$= \prod_{i=1}^{n} ce^{-x_i^4} \quad (735)$$

$$= e^{nln(c) - \sum_{i=1}^{n} x_i^4} \quad (736)$$

$$(737)$$

Plugging this in for $f(x^n)$ in the above inequality and using algebraic manipulation gives:

$$n(ln(c) + (h-\epsilon)ln(2)) \geq \sum_{i=1}^{n} x_i^4 \geq n(ln(c) + (h+\epsilon)ln(2))$$

So the shape of the typcial set is the shell of a 4-norm ball $\{x^n : ||x^n||_4 \in (n(ln(c) + (h \pm \epsilon)ln(2)))^{1/4}\}$ .

**8.11**

11) **Non ergodic Gaussian process.**
Consider a constant signal $V$ in the presence of iid observational noise $\{Z_i\}$. Thus $X_i = V + Z_i$, where $V \sim N(0, S)$, and $Z_i$ are iid $\sim N(0, N)$. Assume $V$ and $\{Z_i\}$ are independent.
  a) Is $\{X_i\}$ stationary?
  b) Find $\lim_{n\longrightarrow\infty} \frac{1}{n}\sum_{i=1}^{n} X_i$. Is the limit random?

11) *Nonergodic Gaussian process*
  a) Yes. $EX_i = EV + Z_i = 0$ for all $i$, and

$$EX_iX_j = E(V+Z_i)(V+Z_j) = \begin{cases} S, & i = j \\ S+N. & i \neq j \end{cases} \quad (738)$$

Since $X_i$ is Gaussian distributed it is completely characterized by its first and second moments. Since the moments are stationary, $X_i$ is wide sense stationary, which for a Gaussian distribution implies that $X_i$ is stationary.

  b)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(Z_i + V) \quad (739)$$

$$= V + \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Z_i \quad (740)$$

$$= V + EZ_i(\text{by the strong law of large numbers}) \quad (741)$$

$$= V \quad (742)$$

The limit is a random variable $\mathcal{N}(0, S)$.