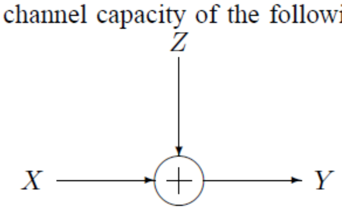
Problem Solving Session 5

7.2

2) **An additive noise channel.** Find the channel capacity of the following discrete memoryless channel:



where $\Pr\{Z=0\} = \Pr\{Z=a\} = \frac{1}{2}$. The alphabet for x is $\mathbf{X} = \{0,1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

2) *A sum channel.*

$$Y = X + Z \qquad X \in \{0,1\}, \quad Z \in \{0,a\}$$

We have to distinguish various cases depending on the values of a.

- $a = 0$ In this case, $Y = X$, and $\max I(X;Y) = \max H(X) = 1$. Hence the capacity is 1 bit per transmission.
- $a \neq 0, \pm 1$ In this case, Y has four possible values $0, 1, a$ and $1+a$. Knowing Y, we know the X which was sent, and hence $H(X|Y) = 0$. Hence $\max I(X;Y) = \max H(X) = 1$, achieved for an uniform distribution on the input X.
- $a = 1$ In this case Y has three possible output values, $0, 1$ and 2, and the channel is identical to the binary erasure channel discussed in class, with $a = 1/2$. As derived in class, the capacity of this channel is $1 - a = 1/2$ bit per transmission.
- $a = -1$ This is similar to the case when $a = 1$ and the capacity here is also $1/2$ bit per transmission.

7.3

3) **Channels with memory have higher capacity.** Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0,1\}$.
Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that $Z_1, Z_2, \ldots, Z_n$ are not necessarily independent. Assume that Z^n is independent of the input X^n. Let $C = 1 - H(p, 1-p)$. Show that $\max_{p(x_1, x_2, \ldots, x_n)} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq nC$.

3) *Channels with memory have a higher capacity.*

$$Y_i = X_i \oplus Z_i,$$

where

$$Z_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

and Z_i are not independent.

$$\begin{aligned} & I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \\ = \; & H(X_1, X_2, \ldots, X_n) - H(X_1, X_2, \ldots, X_n | Y_1, Y_2, \ldots, Y_n) \\ = \; & H(X_1, X_2, \ldots, X_n) - H(Z_1, Z_2, \ldots, Z_n | Y_1, Y_2, \ldots, Y_n) \\ \geq \; & H(X_1, X_2, \ldots, X_n) - H(Z_1, Z_2, \ldots, Z_n) \\ \geq \; & H(X_1, X_2, \ldots, X_n) - \sum H(Z_i) \\ = \; & H(X_1, X_2, \ldots, X_n) - nH(p) \\ = \; & n - nH(p), \end{aligned}$$

if $X_1, X_2, \ldots, X_n$ are chosen i.i.d. $\sim$ Bern($\frac{1}{2}$). The capacity of the channel with memory over n uses of the channel is

$$\begin{aligned} nC^{(n)} &= \max_{p(x_1, x_2, \ldots, x_n)} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \\ &\geq I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n)_{p(x_1, x_2, \ldots, x_n) = \text{Bern}(\frac{1}{2})} \\ &\geq n(1 - H(p)) \\ &= nC. \end{aligned}$$

Hence channels with memory have higher capacity. The intuitive explanation for this result is that the correlation between the noise decreases the effective noise; one could use the information from the past samples of the noise to combat the present noise.

7.4

4) **Channel capacity.** Consider the discrete memoryless channel $Y = X + Z \pmod{11}$, where

$$Z = \begin{pmatrix} 1, & 2, & 3 \\ 1/3, & 1/3, & 1/3 \end{pmatrix}$$

and $X \in \{0, 1, \ldots, 10\}$. Assume that Z is independent of X.

a) Find the capacity.
b) What is the maximizing $p^*(x)$?

4) *Channel capacity.*

$$Y = X + Z (\text{mod } 11)$$

where

$$Z = \begin{cases} 1 & \text{with probability} 1/3 \\ 2 & \text{with probability} 1/3 \\ 3 & \text{with probability} 1/3 \end{cases}$$

In this case,

$$H(Y|X) = H(Z|X) = H(Z) = \log 3,$$

independent of the distribution of X, and hence the capacity of the channel is

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) \\ &= \max_{p(x)} H(Y) - H(Y|X) \\ &= \max_{p(x)} H(Y) - \log 3 \\ &= \log 11 - \log 3, \end{aligned}$$

which is attained when Y has a uniform distribution, which occurs (by symmetry) when X has a uniform distribution.

a) The capacity of the channel is $\log \frac{11}{3}$ bits/transmission.

b) The capacity is achieved by an uniform distribution on the inputs. $p(X = i) = \frac{1}{11}$ for $i = 0, 1, \ldots, 10$.