Problem Solving Session 3

1. **Huffman Codes and "Slice" Questions**
2. **Huffman codes and Shannon codes**
3. **Fano codes (On your own)**
4. **Optimality of Huffman Codes**

5.

Coin weighing. Suppose one has n coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it may be either heavier or lighter than the other coins. The coins are to be weighed by a balance.

a) Find an upper bound on the number of coins n so that k weighings will find the counterfeit coin (if any) and correctly declare it to be heavier or lighter.

b) *(Difficult)* What is the coin weighing strategy for $k = 3$ weighings and 12 coins?

a) For n coins, there are $2n+1$ possible situations or "states".

- One of the n coins is heavier.
- One of the n coins is lighter.
- They are all of equal weight.

Each weighing has three possible outcomes - equal, left pan heavier or right pan heavier. Hence with k weighings, there are 3^k possible outcomes and hence we can distinguish between at most 3^k different "states". Hence $2n+1 \leq 3^k$ or $n \leq (3^k-1)/2$.

Looking at it from an information theoretic viewpoint, each weighing gives at most $\log_2 3$ bits of information. There are $2n+1$ possible "states", with a maximum entropy of $\log_2(2n+1)$ bits. Hence in this situation, one would require at least $\log_2(2n+1)/\log_2 3$ weighings to extract enough information for determination of the odd coin, which gives the same result as above.

b) There are many solutions to this problem. We will give one which is based on the ternary number system.

We may express the numbers $\{-12,-11,\ldots,-1,0,1,\ldots,12\}$ in a ternary number system with alphabet $\{-1,0,1\}$. For example, the number 8 is (-1,0,1) where $-1\times 3^0+0\times 3^1+1\times 3^2=8$. We form the matrix with the representation of the positive numbers as its columns.

	1	2	3	4	5	6	7	8	9	10	11	12	
3^0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	$\Sigma_1=0$
3^1	0	1	1	1	-1	-1	-1	0	0	0	1	1	$\Sigma_2=2$
3^2	0	0	0	0	1	1	1	1	1	1	1	1	$\Sigma_3=8$

Note that the row sums are not all zero. We can negate some columns to make the row sums zero. For example, negating columns 7,9,11 and 12, we obtain

	1	2	3	4	5	6	7	8	9	10	11	12	
3^0	1	-1	0	1	-1	0	-1	-1	0	1	1	0	$\Sigma_1=0$
3^1	0	1	1	1	-1	-1	1	0	0	0	-1	-1	$\Sigma_2=0$
3^2	0	0	0	0	1	1	-1	1	-1	1	-1	-1	$\Sigma_3=0$

Now place the coins on the balance according to the following rule: For weighing $\#i$, place coin n

- On left pan, if $n_i=-1$.
- Aside, if $n_i=0$.
- On right pan, if $n_i=1$.

The outcome of the three weighings will find the odd coin if any and tell whether it is heavy or light. The result of each weighing is 0 if both pans are equal, -1 if the left pan is heavier, and 1 if the right pan is heavier. Then the three weighings give the ternary expansion of the index of the odd coin. If the expansion is the same as the expansion in the matrix, it indicates that the coin is heavier. If the expansion is of the opposite sign, the coin is lighter. For example, (0,-1,-1) indicates $(0)3^0+(-1)3+(-1)3^2=-12$, hence coin #12 is heavy, (1,0,-1) indicates #8 is light, (0,0,0) indicates no odd coin.

Why does this scheme work? It is a single error correcting Hamming code for the ternary alphabet (discussed in Section 8.11 in the book). Here are some details.

First note a few properties of the matrix above that was used for the scheme. All the columns are distinct and no two columns add to (0,0,0). Also if any coin is heavier, it will produce the sequence of weighings that matches its column in the matrix. If it is lighter, it produces the negative of its column as a sequence of weighings. Combining all these facts, we can see that any single odd coin will produce a unique sequence of weighings, and that the coin can be determined from the sequence.

One of the questions that many of you had whether the bound derived in part (a) was actually achievable. For example, can one distinguish 13 coins in 3 weighings? No, not with a scheme like the one above. Yes, under the assumptions under which the bound was derived. The bound did not prohibit the division of coins into halves, neither did it disallow the existence of another coin known to be normal. Under both these conditions, it is possible to find the odd coin of 13 coins in 3 weighings. You could try modifying the above scheme to these cases.