Problem 1. Let $X$ be a discrete RV. Show that the entropy of a function of $X$ is less than or equal to the entropy of $X$.

Problem 2. A function $f: 2^\Omega \to \mathbb{R}$ is said to be submodular if $\forall S \subseteq S' \subseteq \Omega$ and $\forall z \notin S'$, one has

$$f(S \cup \{z\}) - f(S) \geq f(S' \cup \{z\}) - f(S').$$

1. Show that entropy and mutual information are submodular functions.
2. There exist other, equivalent definitions of submodularity that are not an easy modification of the formula above. Find one such characterization and see what it reveals to you about entropy and mutual information. This is an open ended problem that requires you to read a bit more about submodular functions.

Problem 3. Suppose that one has $n$ coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it may be either heavier or lighter than the other coins. The coins are to be weighed by a balance. Find an upper bound on the number of coins $n$ so that $k$ weighings will find the counterfeit coin (if any) and correctly declare it to be heavier/lighter. Try to use information-theoretic arguments.

Problem 4. Three squares have average area $\bar{a} = 100\text{m}^2$. The average of the lengths of their sides is $\bar{l} = 10\text{m}$. What can be said about the area of the largest square?

Problem 5. The Rényi entropy of order $\alpha \geq 0, \alpha \neq 1$ of a discrete RV $X$ supported on a set of cardinality $M$ is defined as

$$H_{\alpha}(X) = \frac{1}{1-\alpha} \log \left( \sum_{i} p_i^\alpha \right).$$

1. Show that $H_0 \geq H_1 \geq H_2 \geq H_\infty$. Observe that the subscripts 1 and $\infty$ are to be taken in the sense of a limit (i.e., $\alpha \to 1, \alpha \to \infty$, respectively).
2. Show that Rényi entropy is non-negative, and that it is concave for $\alpha \leq 1$. 