ECE 561 **Spring 2014**

March 10, 2014

EXAM 1: 7:00–8:30 PM (*90 pts total*)

You are allowed to use *one* sheet of notes (8.5in x 11in, 2-sided). Otherwise the exam is closed book.

1. (35 pts) *Binary Detection with Discrete Observations.* Consider the binary detection problem with $\mathcal{Y} = \{0, 1, 2, 3\}$ and
$$p_1(y) = \frac{y}{6} \quad \text{and} \quad p_0(y) = \frac{1}{4}, \quad y \in \mathcal{Y}$$
   (a) (2 pts) Find a Bayes rule for equal priors and uniform costs
   (b) (4 pts) Find the Bayes risk for the rule in part (a)
   (c) (5 pts) Assuming uniform costs, plot the Bayes risk lines for the following two tests:
$$\delta^{(1)}(y) = \begin{cases} 1 & \text{if } y = 2, 3 \\ 0 & \text{if } y = 0, 1 \end{cases} \qquad \delta^{(2)}(y) = \begin{cases} 1 & \text{if } y = 3 \\ 0 & \text{if } y = 0, 1, 2 \end{cases}$$
   (d) (10 pts) Find a minimax rule for uniform costs.
   (e) (10 pts) Find a Neyman-Pearson rule for level $\alpha = \frac{1}{3}$.
   (f) (4 pts) Find $\mathrm{P_D}$ for the rule in part (e).

2. (25 pts) *Composite Hypothesis Testing.* Consider the hypothesis testing problem:
$$H_0 : \theta = 0$$
$$H_1 : \theta > 0$$
   where
$$p_\theta(\boldsymbol{y}) = \frac{1}{2\pi} \exp\left[-\frac{(y_1 - \theta)^2 + (y_2 - \theta^2)^2}{2}\right]$$
   (a) (10 pts) Is there a UMP test between H_0 and H_1? If so, find it (for a level of $\alpha \in (0, 1)$. If not, explain clearly why not.
   (b) (10 pts) Find an α-level locally optimum test for this problem, for $\alpha \in (0, 1)$.
   (c) (5 pts) Find $\mathrm{P_D}$ as a function of α and θ for the locally optimum test.

3. (30 pts) *Shorts*

   (a) (10 pts) Consider a binary hypothesis testing problem with uniform costs. You are only partially specified the Bayes risk curve for the problem:
$$V(\pi_0) = \min\{\pi_0,\ 0.3 - (\pi_0 - 0.5)^2\} \quad \text{for } 0 \le \pi_0 \le 0.6.$$
   Use this information to find a minimax rule for this problem and the corresponding minimax risk.

(b) (10 pts) Consider the composite hypothesis testing problem:

$$H_0: \ \theta = 0$$
$$H_1: \ \theta \in \{+1, -1\}$$

where

$$p_\theta(y) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(y-\theta)^2}{2}\right]$$

Find the structure of the GLRT solution. Simplify as much as possible.

(c) (10 pts) Consider the Bayesian binary hypothesis testing problem

$$H_0: \ Y \sim p_0$$
$$H_1: \ Y \sim p_1$$

with prior π_0 and uniform costs. The decision maker does not have access to Y, but observes the random variable,

$$U = \begin{cases} 1 & \text{if } L(Y) \geq \frac{\pi_0}{1-\pi_0} \\ 0 & \text{otherwise} \end{cases}$$

where $L(Y) = \frac{p_1(Y)}{p_0(Y)}$.

Show that a Bayes rule based on U is equivalent to a Bayes rule based on Y.