(Notes on a portion of Sedar's lecture. See Meyn and Tweedie for more detail. -BH)
Let X be a time-homogeneous, discrete-time Markov process with a discrete state space $\mathcal{S}$ and one-step transition probability matrix P, relative to some filtration $\boldsymbol{\mathcal{F}}$. Given a function V on $\mathcal{S}$, let PV denote the new function defined by

$$PV(x) = E[V(X(k+1))|X(k) = x] = \sum_{y \in \mathcal{S}} P_{xy}V(y)$$

Proposition 0.1 *Suppose $V, f,$ and g are nonnegative functions on $\mathcal{S}$ such that $PV \leq V - f + g$ (i.e. $PV(x) \leq V(x) - f(x) + g(x)$ for all $x \in \mathcal{S}$.) Then for any stopping time τ and any state $x \in \mathcal{S}$,*

$$E_x\left[\sum_{k=0}^{\tau-1} f(x(k))\right] \leq V(x) + E_x\left[\sum_{k=0}^{\tau-1} g(X(k))\right] \tag{1}$$

In particular, if the right hand side of (1) is finite, then so is the left hand side.

Proof. Let $M = (M(n) : n \geq 0)$ be the random process defined by $M(0) = V(x)$ and
$M(n) = V(X(n)) + \sum_{k=0}^{n-1}(f(X(k)) - g(X(k))$. Given $N \geq 1$ define

$$\tau^N = \min\{\tau, N, \min\{k : f(X(k)) + g(X(k)) \geq N\}\}$$

Then it can be shown that $(M(n \wedge \tau^N) : n \geq 0)$ is a supermartingale if the initial state of the Markov process is x. In particular, it can be shown, using the properties of the stopping time τ_N and induction on n, that $M(n \wedge \tau^N)$ has a finite mean for all $n \geq 0$ and fixed initial state x. Because τ_N is bounded by N, this supermartingale at time $n = N$ is equal to $M(\tau^N)$. Therefore, $E_x[M(\tau^N)] \leq E_x[M(0)] = V(x)$. Equivalently,

$$E_x\left[V(X(n)) + \sum_{k=0}^{\tau^N-1}(f(X(k)) - g(X(k)))\right] \leq V(x) \tag{2}$$

The term $V(X(n))$ is nonnegative, so it can be dropped from (2) and the resulting inequality is still true. Adding $E_x\left[\sum_{k=0}^{\tau^N-1} g(X(k)))\right]$ to each side of (2) yields [1]

$$E_x\left[\sum_{k=0}^{\tau_N-1} f(x(k))\right] \leq V(x) + E_x\left[\sum_{k=0}^{\tau_N-1} g(X(k))\right] \tag{3}$$

Letting $N \to \infty$ in (3) and applying the monotone convergence theorem to each side yields the desired result.

[1] We can show that the quantity added is, in fact, bounded by N^2, but we really only need to know it is in $[0, +\infty]$ for this step.