ECE 559BH Distributed Network Algorithms **SPRING 2006**
Solutions to PROBLEM SET 2

1. Two martingales associated with a simple branching process
(a) Let $\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n)$. Since each of the Y_n individuals in the n^{th} generation has an average of μ offspring, $E[Y_{n+1}|\mathcal{F}_n] = \mu Y_n$. Thus, $E[G_{n+1}|\mathcal{F}_n] = \frac{\mu Y_n}{\theta^{n+1}} = \frac{\mu}{\theta} G_n$. Taking $\theta = \mu$ makes G a martingale.
(b) Given Y_n, the descendants of each one of the Y_n individuals of the n^{th} generation evolve independently. Each of these subpopulations has an extinction probability of α, and the overall population becomes extinct if and only if all Y_n subpopulations become extinct. Hence, $P[\mathcal{E}|Y_0, \ldots, Y_n] = \alpha^{Y_n}$. Thus M is a Doob martingale. (Doob martingales converge with probability one. Only the limits 0 and 1 are possible in this case, for otherwise the process G_n keeps moving. Thus, either $Y_n \to 0$ or $Y_n \to \infty$.)
(c) $M_0 = \alpha$ with probability one, and $E[M_1] = P^*(\alpha)$, where $P^*(\alpha) = \sum_{k=0}^{\infty} p_k \alpha^k$ and $(p_k : k \geq 0)$ is the probability distribution of the number of offspring of an individual. Hence α satisfies $P^*(\alpha) = \alpha$.

2. Function computation in simple broadcast networks (Lei Ying)
Taking a_2 so large that $(4p(1-p))^{a_2/2} \leq e^{-2}$ yields that $\widetilde{b}_j$ has error probability at most $(4p(1-p))^{(a_2/2)\ln N} \leq \frac{1}{(\ln N)^2}$ for all j. (Ying, slide 11)

Let $\epsilon > 0$ so small that $p + \epsilon < 1/2$. Then if N is large enough, and if a_1 is chosen so large that $(4(p+\epsilon)(1-p-\epsilon))^{a_1/2} \leq e^{-2}$, the fusion center obtains the correct parity of a given subset with error probability at most $1/(a_1 N \ln N)$, in N is sufficiently large. (Ying, slide 13).

Thus, for a given bit i fixed, and a given subset $S_{i,k}$ containing i, the bit $P_{i,k}$ has error probability less than or equal to $1/(a_1 N \ln N) \leq \frac{1}{(\ln N)^2}$ and each of the $a_1 \ln(N-1)$ b_j's also have error probability less than or equal to $\frac{1}{(\ln N)^2}$. Thus, by the union bound, the error probability for $\bar{b}_{i,k}$ is at most $a_1/\ln N$. The final estimate of b_i thus has error probability at most $(4a_1/\ln N)^{(a_1/2)\ln N}$, which is smaller than $\frac{1}{N^2}$ for N large enough. Hence, the probability that at least one of the final bits is in error is less than or equal to $\frac{1}{N}$.

3. A covering problem
(a) Let X_i denote the location of the i^{th} base station. Then $F = f(X_1, \ldots, X_m)$, where f satisfies the Lipschitz condition with constant $(2r-1)$. Thus, by the method of bounded differences based on the Azuma-Hoeffding inequality, $P\{|F - E[F]| \geq \gamma\} \leq 2\exp(-\frac{\gamma^2}{m(2r-1)^2})$.
(b) Using the Possion method and associated bound technique, we compare to the case that the number of stations has a Poisson distribution with mean m. Note that the mean number of stations that cover cell i is $\frac{m(2r-1)}{n}$, unless cell i is near one of the boundaries. If cells 1 and n are covered, then all the other cells within distance r of either boundary are covered. Thus,

$$\begin{aligned} P\{X \geq m\} &\leq 2P\{\text{Poi}(m) \text{ stations is not enough}\} \\ &\leq 2ne^{-m(2r-1)/n} + P\{\text{cell 1 or cell } n \text{ is not covered}\} \\ &\to 0 \quad \text{as } n \to \infty \qquad \text{if } m = \frac{(1+\epsilon)n \ln n}{2r-1} \end{aligned}$$

As to a bound going the other direction, note that if cells differ by $2r-1$ or more then the events that they are covered are independent. Hence,

$$\begin{aligned} P\{X \leq m\} &\leq 2P\{\text{Poi}(m) \text{ stations cover all cells}\} \\ &\leq 2P\{\text{Poi}(m) \text{ stations cover cells } 1 + (2r-1)j, 1 \leq j \leq \frac{n-1}{2r-1}\} \\ &\leq 2\left(1 - e^{-\frac{m(2r-1)}{n}}\right)^{\frac{n-1}{2r-1}} \\ &\leq 2\exp(-e^{-\frac{m(2r-1)}{n}} \cdot \frac{n-1}{2r-1}) \\ &\to 0 \quad \text{as } n \to \infty \qquad \text{if } m = \frac{(1-\epsilon)n \ln n}{2r-1} \end{aligned}$$

Thus, in conclusion, we can take $g_1(r) = g_2(r) = \frac{1}{2r-1}$.

4. Doob decomposition

Parts (a) and (b) are solved in reverse order. (b) Assume the decomposition exists. We shall show it is unique. First, $E[X_0] = E[M_0 + B_0] = B_0$. Therefore, $B_0 = E[X_0]$ and $M_0 = X_0 - B_0$. Similarly, for $k \geq 0$, $X_{k+1} - X_k = B_{k+1} - B_k + M_{k+1} - M_k$. Since B is predictable, $B_{k+1} - B_k$ is $\mathcal{F}_k$ measurable, so that $E[B_{k+1} - B_k|\mathcal{F}_k] = B_{k+1} - B_k$. Since M is a martingale, $E[M_{k+1} - M_k|\mathcal{F}_k] = 0$. Therefore, $E[X_{k+1} - X_k|\mathcal{F}_k] = B_{k+1} - B_k$. Thus, for $n \geq 0$,

$$\begin{aligned} B_n &= E[X_0] + \sum_{k=0}^{n-1} E[X_{k+1} - X_k|\mathcal{F}_k] \qquad \text{and} \\ M_n &= X_0 - E[X_0] + \sum_{k=0}^{n-1} \left(X_{k+1} - X_k - E[X_{k+1} - X_k|\mathcal{F}_k]\right). \end{aligned}$$

Thus, M and B are uniquely determined by X and $\boldsymbol{\mathcal{F}}$.
(a) The decomposition exits. It is given by the sequences M and B found in part (b).

5. On uniform integrability

(a) Use the small set characterization of u.i. First, $\sup_i(E[|X_i|]+E[|Y_i|]) \leq (\sup_i E[|X_i|])+(\sup_i E[|Y_i|]) < \infty$ Second, given $\epsilon > 0$, there exists δ so small that $E[|X_i|I_A] \leq \frac{\epsilon}{2}$ and $E[|Y_i|I_A] \leq \frac{\epsilon}{2}$ for all i, whenever A is an event with $P\{A\} \leq \epsilon$. Therefore $E[(|X_i + Y_i|I_A] \leq E[|X_i|I_A] + E[|Y_i|I_A] \leq \epsilon$ for all i, whenever A is an event with $P\{A\} \leq \epsilon$. Thus, $(X_i + Y_i : i \in I)$ is u.i.
(b) Suppose $(X_i : i \in I)$ is u.i., which by definition means that $\lim_{c\to\infty} K(c) = 0$, were $K(c) = \sup_{i\in I} E[|X_i|I_{\{|X_i|\geq c\}}]$. Let c_n be a sequence of positive numbers monotonically converging to $+\infty$ so that $K(c_n) \leq 2^{-n}$. Let $\varphi(u) = \sum_{n=1}^{\infty}(u - c_n)_+$. The sum is well defined, because for any u, only finitely many terms are nonzero. The function φ is convex and increasing because each term in the sum is. The slope of $\varphi(u)$ is at least n on the interval (c_n, ∞) for all n, so that $\lim_{u\to\infty} \frac{\varphi(u)}{u} = +\infty$. Also, for any $i \in I$ and c, $E[(|X_i| - c)_+] = E[(|X_i| - c)I_{\{|X_i|\geq c\}}] \leq K(c)$.
Therefore, $E[\varphi(|X_i|)] \leq \sum_{n=1}^{\infty} E[(|X_i| - c_n)_+] \leq \sum_{n=1}^{\infty} 2^{-n} \leq 1$. Thus, the function φ has the required properties.

The converse is now proved. Suppose φ exists with the desired properties, and let $\epsilon > 0$. Select c_o so large that $u \geq c_o$ implies that $u \leq \epsilon\varphi(u)$. Then, for all $c \geq c_o$,

$$E[|X_i|I_{\{|X_i|\geq c\}}] \leq \epsilon E[\varphi(|X_i|)I_{\{|X_i|\geq c\}}] \leq \epsilon E[\varphi(|X_i|)] \leq \epsilon K$$

Since this inequality holds for all $i \in I$, and ϵ is arbitrary, it follows that $(X_i : i \in I)$ is u.i.