ECE 559BH Distributed Network Algorithms **SPRING 2006**
SOLUTIONS TO PROBLEM SET 1

3. Large deviations for the coupon collectors problem
Let M be a Possion distributed random variable with mean $(1-\epsilon)n\ln n$. Then using problem 2, and the fact that the probability of having a complete collection is monotone increasing in the number of coupons collected,

$$\begin{aligned} P\{M \text{ coupons is enough }\} &\geq P[M \text{ coupons is enough } |M \geq (1-\epsilon)n\ln n]P[M \geq (1-\epsilon)n\ln n] \\ &\geq P\{X < (1-\epsilon)n\ln n\} \times \frac{1}{2} \end{aligned}$$

Thus,

$$\begin{aligned} P\{X < (1-\epsilon)n\ln n\} &\leq 2P\{M \text{ coupons is enough }\} \\ &= 2(1-e^{-(1-\epsilon)\ln n)})^n \\ &= 2(1-n^{-(1-\epsilon)})^n \\ &\leq 2\exp(-n^{-(1-\epsilon)} \times n) = 2\exp(-n^{\epsilon}) \end{aligned}$$

Similarly, if Z is a Poission distributed random variable with mean $(1+\epsilon)n\ln n$, then

$$\begin{aligned} P\{X > (1+\epsilon)n\ln n\} &\leq 2P\{Z \text{ coupons is not enough }\} \\ &= 2[1-(1-e^{-(1+\epsilon)\ln n)})^n] \\ &= 2[1-(1-n^{-(1+\epsilon)})^n] \\ &\leq 2[1-(1-n^{-(1+\epsilon)} \times n)] = 2n^{-\epsilon} \end{aligned}$$