Truthful and Efficient Combinatorial Auctions

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Topics

1. Introduction to combinatorial auctions
2. Definitions
3. the Generalized Vickrey Auction (GVA)
4. Single-minded bidders
5. the truthful payement scheme for greedy allocation
6. Complex bidders

What is combinatorial auction?

an auction that multiple goods are available and bidders can post bids for subsets of the goods. We consider the case of a single seller (auctioneer) and multiple buyers (bidders), one-stage and sealed-bid CAs.

A bid may specify

Complementarity a whole set of items may worth differently then the sum of the value of individual items in the same set to a particular buyer.

Substitutability the ownership of one item will affect the value of another item to a particular buyer.

We use the bidding type which specifies only complementarity. We also assume that there no externality is allowed. In other word, all bidders are independent and bid for their own interest.

THE GOAL OF AN AUCTION

Maximize revenue maximize the income for the seller, including the truthful of bids and allocation algorithm.

Efficient no further trade among the buyers can improve the situation of some trader without hurting any of them. The allocation is done centrally by the auctioneer.

Definitions:

P set of n bidders

G set of k goods

allocation a partial function $a : G \rightarrow P', P' = P \cup \{unallocated\}$

$\mathcal{O}$ set of all allocations $\mathcal{O} = P'^G$

type a bidder's evaluation of goods, a function $e : 2^G \rightarrow \Re_+$

Θ set of all bidder types (bidding message types) $\Theta = \Re_+^{2^G}$

t true type of a bidder $t \in \Theta$

d declared type (bidder may lie) $d \in \Theta$

T a vector of true types $T =< t_1, \ldots, t_n >$ of n bidders

D a vector of declared types $D =< d_1, \ldots, d_n >$ of n bidders

$\mathcal{P}$ a vector of n payments. $\mathcal{P} \in \Re_+^n$

Definition 1 (Utility)**.** For a bidder of type t, of bundles $s \subseteq G$ and payment x, the utility u is

$$u = t(s) - x$$

Utility is a measurement of the gain of a bidder in an auction.

Definition 2 (a direct mechanism for CAs)**.** For each declared type vector D, a mechanism is a tuple of $< f, p >$ denoted by

1. an allocation algorithm f picks an allocation $a = f(D)$.
2. an payment scheme p determines a payment vector $P = p(D)$ where $p_i(D)$ denotes the payment by bidder i to the auctioneer.

Some results from a mechanism:

g_i is the set of goods obtained by bidder i,

$$g_i(D) = f(D)^{-1}(i)$$

Remember $f(D) : G \to P'$.

u_i is the utility of a bidder i from this mechanism,

$$u_i = t(g_i(D)) - p_i(D)$$

Remember $t : G \to \Re_+$.

Definition 3 (Truthful)**.** A mechanism $< f, p >$ is truthful if and only if for every $i \in P, t \in \Theta$ and any vector D of declarations, if D^i is the vector obtained from D by replacing the ith coordinate d_i by t, then

$$t(g_i(D^i)) - p_i(D^i)) \geq t(g_i(D)) - p_i(D)$$

A truthful mechanism increases the utility for a bidder if s/he tells the true evaluation of goods during the bidding.

GVA is a generalization of the second price auctions of Vickrey[1961] by Clarke[1971] and Groves[1973]. The second price auction uses the mechanism that the winner pays the second highest price rather than his/her declared price.

Definition 4 (GVA)**.** Given a vector D of declarations, the GVA defines the allocation and Clarke's payment policy as follows:

$$f(D) = \mathrm{argmax}_{a \in \mathcal{O}} \sum_{i=1}^{n} d_i(a^{-1}(i)) \tag{1}$$

$$p_j(D) = -\sum_{i=1, i \neq j}^{n} d_i(g_i(D)) + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z)) \tag{2}$$

where Z is a vector of declared value, $z_i = d_i$ for $i \neq j$, $z_i(s) = 0$ for any set $s \subseteq G$.

The allocation algorithm maximizes the sum of declared values to a certain allocation. Each bidder receives the amount that equals the sum of the declared values of all other bidders and pays the auctioneer the sum of such valuation that would have been obtained if s/he had not participated in the auction.

Theorem 1. *The GVA auction is a truthful mechanism.*

Proof. Assume a bidder j tells the truth instead of lying in delaration D. D^j is the new declaration where $D_i = D_i^j$ for any $i \neq j$ and $D_j^j = t$. We have

$$t(g_j(D^j)) - p_j(D^j) = t(g_j(D^j)) + \sum_{i=1, i\neq j}^{n} d_i(g_i(D^j)) - \sum_{i=1}^{n} d_i(g_i(Z))$$

$$= d_j^j(g_j(D^j)) + \sum_{i=1, i\neq j}^{n} d_i^j(g_i(D^j)) - \sum_{i=1}^{n} d_i(g_i(Z))$$

Think of $g_i(D)$ as another allocation $a(D^j)^{-1}(i)$, because $g_i(D^j)$ maximize the allocation for D^j, we have

$$\geq \sum_{i=1}^{n} d_i^j(g_i(D)) - \sum_{i=1}^{n} d_i(g_i(Z)) = t(g_j(D)) - p_j(D)$$

$\square$

One can also show that the utility of a truthful bidder in GVA is nonnegative.

Problems

1. The size of the set $\mathcal{O}$ is n^k exponential in k.
2. the size of the set Θ is $|cost|^{2^k}$ doubly exponential in k where $cost$ is a finit subset of $\Re_+$ including all applicable values.

To avoid the exponential length of messages, we simplify our analysis on single-minded bidders.

Definition 5 (Single-Minded Bidders)**.** Bidder i is single-minded if and only if there is a set $s \subseteq G$ of goods and a value $v \in \Re_+$ such that its type t can be described as $t(s') = v$ if $s \subseteq s'$ and $t(s') = 0$ otherwise.

Single-minded bidders only interest in one exact subset of goods. Even they are granted part of the subset, they will evaluate it as zero. SMB types can be described by $< s, v >$. The size of the set of types is reduced to $|cost|^k$.

Theorem 2. *The problem to find an allocation for GVA is NP-hard in k+n even under the assumption of single-minded bidders.*

To solve the problem, we propose *greedy allocation* algorithms to approximate the efficient solution.

Conventions for greedy allocation algorithms for SMB:

A bid $b =< s, a >$ where $s \subseteq G, a \in \Re_+$. Use $s(b)$ to denote the subset information in bid b. $a(b)$ is the declared amount in b. Define two bids b and b' is *conflict* if $s(b) \cap s(b') \neq \emptyset$. Define the l-norm of a bid b is $\frac{a(b)}{|s(b)|^l}$.

A general greedy algorithm consists of

1. Sorting the bid by some criterion. The common way is by its l-norm for some particular chosen l.

2. Generating an allocation by picking the non-conflict bid in order.

The greedy approach usually evaluates the average value in a bid and the bids sorted in a higher rank have a higher priority to get picked up.

Theorem 3. *The greedy allocation scheme with $1/2$-norm approximates the optimal allocation within a factor of $\sqrt{k}$.*

Theorem 4. *The greedy allocation with Clarke's payment scheme does not make a truthful mechanism even for single-minded bidders.*

Proof. Example shows that a bidder may have incentive to cheat to increase utility. Two goods a and b and three bidders Red, Green and Blue. Red bids 10 for a, Green bids 19 for $\{a, b\}$ and Blue bids 8 for b. Allocation of greedy algorithm is $f(D)(a) = Red$ and $f(D)(b) = Blue$. By Clarke's payment scheme, Red pays 19 to Green and -8 from Blue and total will be 11. Utility of Red is -1. However, if Red cheats by bidding $< 9, a >$, the allocation will go to Green. Then, Red pays zero for nothing. The utility is 0 better than -1. $\square$

One of the important result of this paper is the sufficient condition for a truthful mechanism for SMB.

Theorem 5. *If a mechanism satisfies Exactness, Monotonicity, Participation and Critical, then it is a truthful mechanism.*

Exactness Either $g_j = s$ or $g_j = \emptyset$.

The allocation among SMB should be exact the subset of goods s/he wants or nothing.

Monotonicity If $s \subseteq g_j, s' \subseteq s, v' \geq v$ then $s' \subseteq g'_j$.

As a bidder increases the value for the same bundle or reduce the size of the bundle requested, his or her bid will be granted if the original bit is granted.

Lemma 1. *In a mechanism satisfying Exactness and Monotonicity, for a bidder* j*, there exists a critical value* v_c *such that* $\forall v, v < v_c$ *then* $g_j = \emptyset$ *and* $\forall v, v > v_c$ *then* $g_j = s$*.*

The critical value tells the uniqueness of determination factor for a bidder to win a bid.

Critical If $s \subseteq g_j$ then $p_j = v_c$ where v_c is the critical value for bidder j.

The payment should not depend on the amount of a single bid, instead on all other bids. Since all bidders are independent, the payment scheme are more objective and insensitive to a single declaration.

Participation If $s \nsubseteq g_j$ then $p_j = 0$.

The utility of an unsatisfied bidder is zero.

Then, we prove the theorem.

Proof. Suppose bidder j's true type is $< s, v >$. Show that j cannot get positive utility by declaring $< s', v' >$. First show that in the case that the new bid is not granted, the utility is zero. Then consider the case that the new bid is granted then $g_j = s'$. If $s \nsubseteq s'$, the true valuation according to SMB is zero. Thus, the utility will not be positive. Then consider $s \subseteq s'$, by Monotonicity and Critical, the bidder won't lose by just declaring $< s, v' >$ and again by Critical, the payments for declaring $< s, v >$ and $< s, v' >$ are the same. Therefore, the bidder has no incentive to lie. $\square$

Definition 6 (Greedy payment scheme)**.** Denote $c(j)$ be the norm j and $n(j)$ be the first bid following j in the sorted order that conflicts with j. Then, j pays zero if his bid is denied or if there is no bid $n(j)$. If j's bid is granted and there is an $n(j)$, j pays $|s| \times c(n(j))$.

Note that if j is granted, $n(j)$ would have been granted if there was no j. Actually this payment scheme makes j pay at the rate of $n(j)$.

Theorem 6. *The payment scheme and greedy allocation compose a truthful mechanism for SMB.*

Proof. Trivially Participation is satisfied by the payment scheme. Monotonicity and Exactness are satisfied by the greedy allocation since the increase of v and/or decrease of s will increase the norm causing the bid will rank higher. Critical is satisfied by assigning the payment according to the rate of nearest conflicting bid because the bidder would have lost the bid if his/her bidding rate is lower than the nearest conflicting one. $\square$

SMB is too restrictive in realistic. Complex bidders should be considered. One may consider a complex bidder as a superset of SMBs and restrict the set of bids so that the message is still single exponentially with k. Therefore, one can extend greedy allocation to the case of complex bidders. One important result shows that

Theorem 7. *No payment scheme makes the greedy allocation a truthful mechanism for complex bidders.*

The reason behind is that a bidder may manipulate his/her multiple bids to try to get high utility on one bid even bearing the cost of losing utility on others. As long as the totally utility is positive, the bidder still gains. Actually, such generalization of greedy allocation to complex bidders violates the independence between SMBs.

References

[1] D. Lehmann et al, *Truth revelation in approximately efficient combinatorial auctions*, 2002