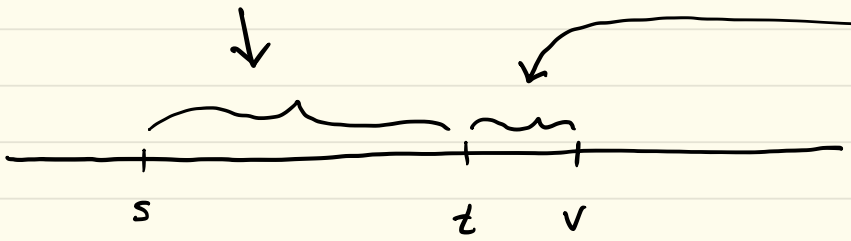
Poisson Process

$N(t)$ a "counting process" if $N(t)$ is number of "events" that occur between time 0 and t, (t is time)

① $N(t)$ is a "Poisson Process" if

- $N(0) = 0$
- $\{N(t)\}$ has independent increments for all $s < t < v$

$N(t) - N(s)$ is independent of $N(v) - N(t)$



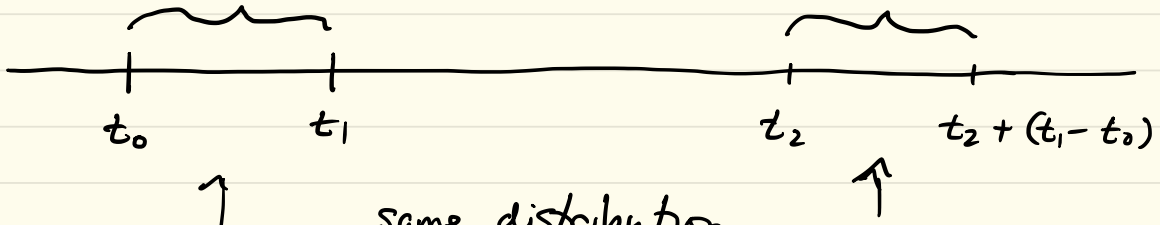
- number of events in any interval of length t has a Poisson distribution with mean λt

λ is rate of process

Equivalent Statements

② $\{N(t)\}$ is a Poisson process if

- $N(0) = 0$
- $N(t)$ has stationary and independent increments



- $P(N(t) \geq 2) = o(t)$
- $P(N(t) = 1) = \lambda \cdot t + o(t)$

f is $o(t)$ if $\lim_{t \to 0} \frac{f(t)}{t} = 0$

③ $N(t)$ counts events when the distribution of time between events is exponential with rate λ

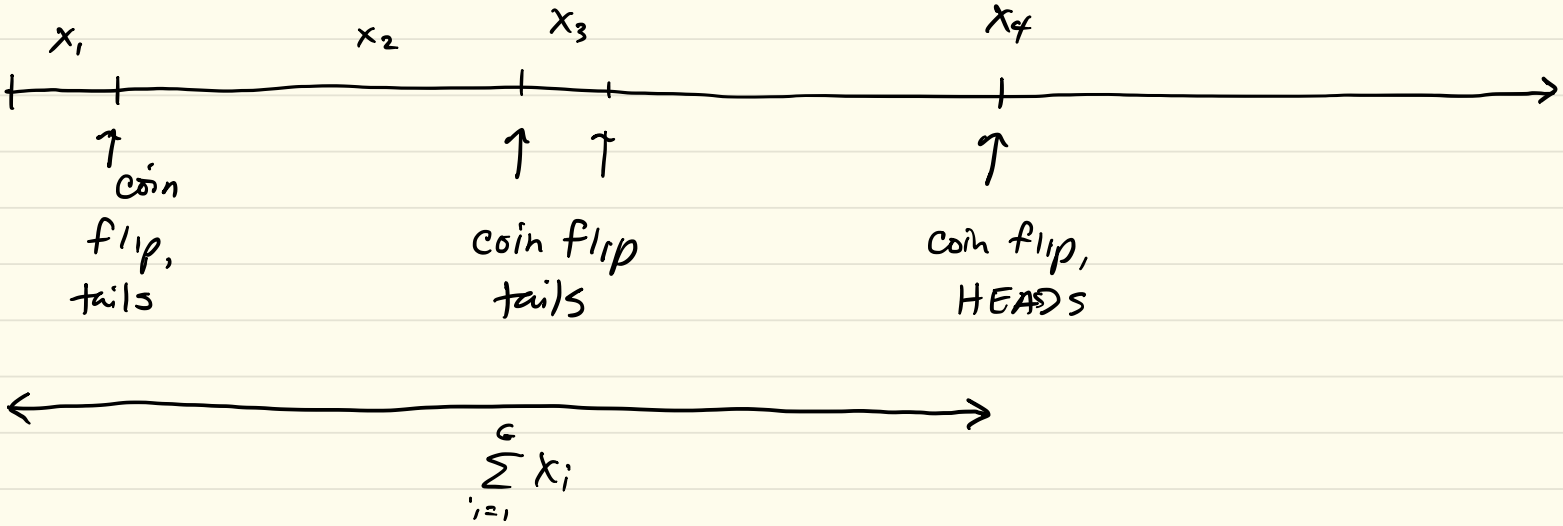
Fun fact we soon use

Let G be a geometric random variable with success parameter p

Let $X_1, X_2, \ldots$ be a sequence of i.i.d exponentials with rate λ

Then $\sum_{i=1}^{G} X_i$ is exponential with rate $p \cdot \lambda$

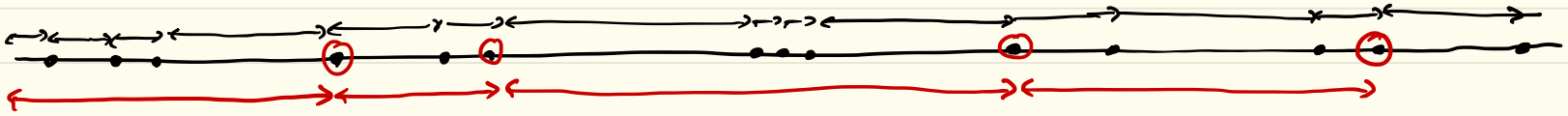
What does this mean?



operationally, with each "original" event, flip coin to decide whether to end or not.

Proof that $\sum_{i=1}^{G} X_i$ is exponential with rate $\lambda \cdot p$

Note that $\sum_{i=1}^{G} X_i$ is inter-arrival time of a counting process that "filters" a Poisson process with rate λ



Reconsider characterization ② of Poisson process

- underlaying PP has stationary & independent increments $\Rightarrow$ filtered process does

Let N be original process and N_p be filtered process

$$P(N_p \geq 2) \leq P(N \geq 2) = o(t)$$

$$P(N_p = 1) = p \cdot P(N = 1) = p \cdot \lambda \cdot t + p \cdot o(t) = \lambda \cdot p \cdot t + o(t)$$

$\Rightarrow N_p$ is Poisson with rate $p \cdot \lambda$

$\Rightarrow$ inter-arrivals are exponential with rate $\lambda \cdot p$



CTMC Transient Solution

We have seen that it is easy to specify a CTMC in terms of the initial probability distribution $\pi(0)$ and the state-transition-rate matrix.

Earlier, we saw that the transient solution of a CTMC is given by $\pi(t) = \pi(0)P(t)$, and we noted that $P(t)$ was difficult to define.

Due to the complexity of the math, we omit the derivation and assert the relationship

$$\frac{d}{dt}P(t) = QP(t) = P(t)Q,$$ where Q is the state transition rate matrix of the Markov chain.

Solving this differential equation in some form is difficult but necessary to compute a transient solution.

Transient Solution Techniques

Solutions to $\frac{d}{dt}P(t) = QP(t)$ can be done in many (dubious) ways*:

- Direct: If the CTMC has N states, one can write N^2 PDEs with N^2 initial conditions and solve N^2 linear equations.
- Laplace transforms: Unstable with multiple "poles"
- Nth order differential equations: Uses determinants and hence is numerically unstable
- Matrix exponentiation: $P(t) = e^{Qt}$, where $e^{Qt} = I + \sum_{n=1}^{\infty} \frac{(Qt)^n}{n!}$.

Matrix exponentiation has some potential. Directly computing e^{Qt} by performing $I + \sum_{n=1}^{\infty} \frac{(Qt)^n}{n!}$ can be expensive and prone to instability.

If the CTMC is irreducible, it is possible to take advantage of the fact that $Q = ADA^{-1}$, where D is a diagonal matrix. Computing e^{Qt} becomes $Ae^{Dt}A^{-1}$, where $e^{Dt} = \text{diag}\left(e^{d_{11}t}, e^{d_{22}t}, \ldots, e^{d_{nn}t}\right)$.

* See C. Moler and C. Van Loan, "Nineteen Dubious Ways to Compute the Exponential of a Matrix," *SIAM Review*, vol. 20, no. 4, pp. 801-836, October 1978.

Standard Uniformization:

On treating a CTMC like a DTMC

how? when?

Given Q for a CTMC

suppose there is λ^* s.t. $\lambda^* \geq |q_{i,i}|$ for all i

Now construct $P = I + \frac{Q}{\lambda^*}$

- off diagonal elements are positive, turned into probabilities
- Diagonal elements are (by addition of 1 on diagonal) probabilities

  EQUAL to 1 minus sum of off-diagonal row probabilities

$$P = I + \frac{Q}{\lambda^*}$$ turns rates into probabilities by proportionality

- if $q_{i,j} = 2q_{i,k}$ then

$$\frac{q_{i,j}}{\lambda^*} = 2\frac{q_{i,k}}{\lambda^*}$$

Question: What is the distribution of the number of time steps a DTMC with

$$P = I + \frac{Q}{\lambda^*}$$

is in state i before changing state?

With each transition there is probability

$$1 + \frac{q_{i,i}}{\lambda^*}$$

of staying

$$\Rightarrow \quad \frac{-q_{i,i}}{\lambda^*}$$ probability of changing state

$\Rightarrow$ Number of consecutive time steps in state i is geometric with success probability

$$\frac{-q_{i,i}}{\lambda^*} = \frac{\sum_{j \neq i} q_{i,j}}{\lambda^*}$$

$\Rightarrow$ Distribution of $\sum_{n=1}^{G} X_i$ when G is that geometric and X_i are exponentials with rate λ^* is

$$\left(\frac{\sum_{j \neq i} q_{i,j}}{\lambda^*} \right) \cdot \lambda^* = \text{holding time rate of CTMC in state } i$$

IMPLICATION

- equivalent simulation of the CTMC

1. $T = 0$, $X(0) =$ starting state, $n = 0$
2. While True
   - advance DTMC one time step
   - sample H from exponential dist. rate λ^*
   - $T \mathrel{+}= H$

i.e., each step of the DTMC is interpretted as having an exponential holding time with rate λ^*
— independent of state

$\Rightarrow$ we can reason about $P(t)$ in terms of underlying uniformized

Over $[0, t]$ number of DTMC transitions is Poisson with rate $\lambda^* \cdot t$

Recall $\lambda^* \geq |q_{i,i}|$ for all i $\qquad P = I + \frac{Q}{\lambda^*}$

and $\pi(t) = \pi(0) P(t)$

Viewed from perspective of uniformized chain,

if we condition on k DTMC transitions,
$P(t) = P^k$, k-fold multiplication of DTMC matrix P

$$P(t) = \sum_{k=0}^{\infty} P(k \text{ DTMC transitions in } [0,t]) \cdot P^k$$

So

$$\pi(t) = \pi(0) \left[\underbrace{\frac{(\lambda t)^0}{0!} \cdot e^{-\lambda t}}_{P(0 \text{ DTMC trans})} \cdot P^0 + \underbrace{\frac{(\lambda t)^1}{1!} \cdot e^{-\lambda t}}_{P(1 \text{ DTMC trans})} \cdot P^1 + \underbrace{\frac{(\lambda t)^2}{2!} e^{-\lambda t}}_{P(2 \text{ DTMC trans})} \cdot P^2 + \cdots \right.$$

$$\pi(0) \cdot \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot P^k$$

N.B. use $\alpha(k)$ to denote state occupancy probability vector at STEP k in DTMC, $\pi(t)$ CTMC vector

$$\alpha(0) = \pi(0)$$

In practice

finite

$$\pi(t) = \alpha(0) \sum_{k=0}^{N_s} \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t} \cdot P^k$$

$$= \alpha(0)\, e^{-\lambda t} \left[1 + \frac{\lambda t}{1!} P + \frac{(\lambda t)^2}{2!} \cdot P^2 + \frac{(\lambda t)^3}{3!} P^3 + \cdots \right.$$

$$= e^{-\lambda t} \left[\alpha(0) + \frac{\lambda t}{1!} \cdot \alpha(0) P + \frac{(\lambda t)^2}{2!} \alpha(0) P^2 + \frac{(\lambda t)^3}{3!} \alpha(0) P^3 \right.$$

use $\alpha(k+1) = \alpha(k) P$

$$= e^{-\lambda t} \left[\alpha(0) + \frac{\lambda t}{1!} \alpha(1) + \frac{(\lambda t)^2}{2!} \alpha(2) + \frac{(\lambda t)^3}{3!} \alpha(3) + \cdots \right.$$

$$= e^{-\lambda t} \cdot \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \cdot \alpha(k)$$

Error bound

$$\tilde{\pi}(t) = e^{-\lambda t} \sum_{k=0}^{N_s} \cdot \frac{(\lambda t)^k}{k!} \cdot \pi(k)$$

each component of $\pi(t)$ is a tad bit too small because we haven't considered more than N_s transitions of the DTMC

for any component i

$$\pi_i(t) - \tilde{\pi}_i(t) = e^{-\lambda t} \cdot \sum_{k=N_s+1}^{\infty} \frac{(\lambda t)^k}{k!} \pi_i(k)$$

$$< e^{-\lambda t} \sum_{k=N_s+1}^{\infty} \left(\frac{\lambda t}{k!}\right)^k$$

since $\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t} = 1$

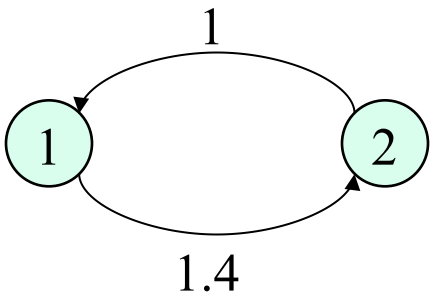
$$e^{-\lambda t} \sum_{k=N_s+1}^{\infty} \frac{(\lambda t)^k}{k!} = 1 - e^{-\lambda t} \sum_{k=0}^{N_s} \frac{(\lambda t)^k}{k!}$$

$$= 1 - e^{-\lambda t} \sum_{k=0}^{N_s} \frac{(\lambda t)^k}{k!}$$

We can dynamically compute as we increase k in sum, and stop when bound $\leq \varepsilon$ as desired

A Simple Example

- Consider the simple CTMC below

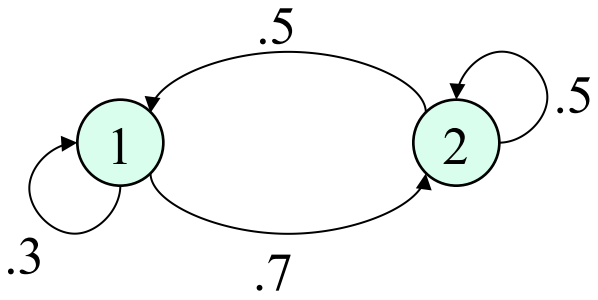


$$Q = \begin{bmatrix} -1.4 & 1.4 \\ 1 & -1 \end{bmatrix}$$

- Note that with $\lambda = 2$, λ exceeds largest magnitude on diagonal
- Define

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1.4/2 & 1.4/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} .3 & .7 \\ .5 & .5 \end{bmatrix}$$

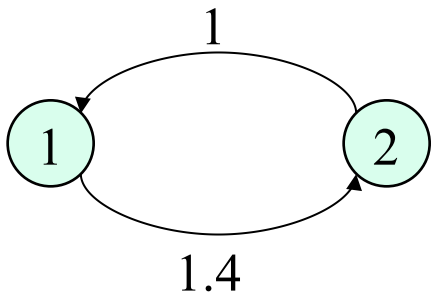
  - Discrete time Markov chain



A Simple Example (continued)

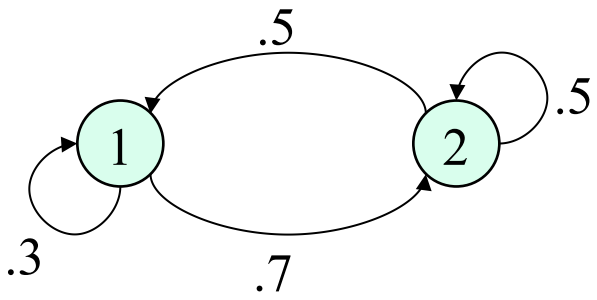
CTMC

$$Q = \begin{bmatrix} -1.4 & 1.4 \\ 1 & -1 \end{bmatrix}$$



- Discrete time Markov chain

$$P = \begin{bmatrix} .3 & .7 \\ .5 & .5 \end{bmatrix}$$



- Make sense?
  - Look at sum of a geometric number of exponentials (geometric with parameter r)
  - Result: exponential with rate $r\lambda$.
  - DTMC steps in state 1 is geometric, p=0.7. Note λ=2 : CTMC rate from state 1 is 0.7*2 = 1.4. Ditto state 2.
  - Matches that for CTMC.

Uniformization cont.

Doing the infinite summation is of course impossible to do numerically, so one would stop after a sufficiently large number.

Uniformization has the following properties:

- Numerically stable
- Easy to calculate the error and stop summing when the error is sufficiently small
- Efficient
- Easy to program
- Error will decrease slowly if there are large differences (orders of magnitude) between rates.