Random Variate Generation

We've seen how to generate UNIFORM random variables

How about other distributions?

Some simple examples

Let U be a uniform $U[0,1]$ random variable

1. Continuous uniform on $[a,b]$

$$f(t) = \frac{1}{b-a} \quad \text{so} \quad F(t) = \int_a^t \frac{1}{b-a} \cdot ds = \frac{t-a}{b-a}$$

Now Define $X = a + (b-a) \cdot U$

$$\Pr\{X < t\} = \Pr\{a + (b-a) \cdot U < t\} = \Pr\left\{U < \frac{t-a}{b-a}\right\}$$

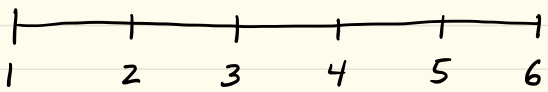
$$= F(t)$$

✓

2. Discrete uniform on $[a, b]$, a, b integers

$$X = \text{integer}\left((b+1-a)\cdot u\right)$$

example $b = 5$, $a = 1$



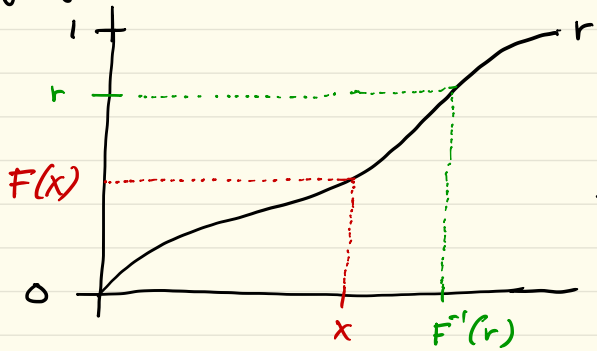
5 values, 5 equi-length regions of number line

$$R_r\{X = i\} = R_r\{\text{integer}((b+1-a)\cdot u) = i\}$$

$$R_r\left\{\frac{i}{(b+1-a)} \le u \le \frac{i+1}{(b+1-a)}\right\} = \frac{1}{b+1-a}$$

INVERSE TRANSFORM METHOD

- extremely general



F is CDF of dist. we wish to sample from

Formally F^{-1} exists because F is monotone and covers $[0,1]$
$\Rightarrow$ for every p there exists unique x with $F^{-1}(p) = x$, namely x s.t. $F(x) = p$

$$F^{-1}(p) = \inf_x \{ F(x) \geq p \}$$

Theorem

Let Y be uniform $U(0,1)$ and define $X = F^{-1}(Y)$
Then X has distribution F

Proof

$$\begin{aligned} \Pr\{X \le t\} &= \Pr\{F^{-1}(Y) \le t\} \\ &= \Pr\{\inf_x \{F(x) \ge Y\} \le t\} \\ &= \Pr\{Y \le F(t)\} \\ &= F(t) \end{aligned}$$

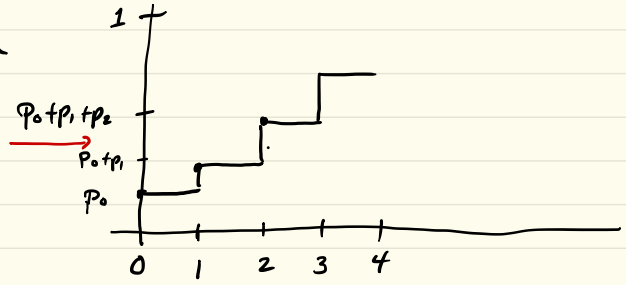
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Any discrete distribution

note: for $p_0 + p_1 < u \le p_0 + p_1 + p_2$

$$\inf_x \{F(x) \ge u\} = 2$$

i.e. we round up.



Continuous needs no rounding

Application

a. Exponential

We know $r = F(x) = 1 - e^{-\lambda x}$

For what x is $F^{-1}(r) = x$?

set up and solve $r = 1 - e^{-\lambda x}$

$$\Rightarrow \quad 1 - r = e^{-\lambda x}$$
$$\Rightarrow \quad \ln(1-r) = \ln\left(e^{-\lambda x}\right)$$
$$\Rightarrow \quad \ln(1-r) = -\lambda \ln(e^{x})$$
$$\Rightarrow \quad \ln(1-r) = -\lambda x$$
$$\Rightarrow \quad -\frac{1}{\lambda}\ln(1-r) = x$$

So sample $x = -\frac{1}{\lambda}\ln(1-u)$

to generate a sample from $\exp(\lambda)$

$$x = \frac{1}{\lambda} \ln(1-u)$$

Aside

IF we have two independent exponentials, with rate λ_1 and λ_2 it is always possible

when $\lambda_1 \ll \lambda_2$ to still get a sample from $\exp(\lambda_1)$ that is larger than a sample from $\exp(\lambda_2)$

BUT imagine you want to COMPARE a system with a "slow" exponential service rate with one that is "faster"

System #1

$$X_{1,i} = \frac{1}{\lambda_1} \ln(1-u_i)$$

System #2

$$X_{2,i} = \frac{1}{\lambda_2} \ln(1-u_i)$$

same u_i

$X_{1,i} > X_{2,i}$

3. Weibull $F(x) = 1 - e^{-(x/\lambda)^k}$ parameters λ, k

Set up $u = 1 - e^{-(x/\lambda)^k}$

$$\Rightarrow 1 - u = e^{-(x/\lambda)^k}$$

$$\Rightarrow \ln(1-u) = \ln\left(e^{-(x/\lambda)^k}\right)$$

$$= -\frac{1}{\lambda^k} \cdot \ln\left(e^{x^k}\right)$$

$$= -\frac{1}{\lambda^k} x^k$$

$$x = \left(-\lambda^k \ln(1-u)\right)^{1/k}$$

$$\Rightarrow = -\lambda \ln(1-u)^{1/k}$$

4. Geometric $\quad F(k) = 1 - (1-p)^{k+1}$

$$F^{-1}(y) = \min_k \{ F(k) \geq y \} = \min_k \{ 1 - (1-p)^{k+1} \geq y \}$$

So given $u \sim U[0,1]$, sample

$$F^{-1}(u) = \min_k \{ 1 - (1-p)^{k+1} \geq u \}$$

Observe that

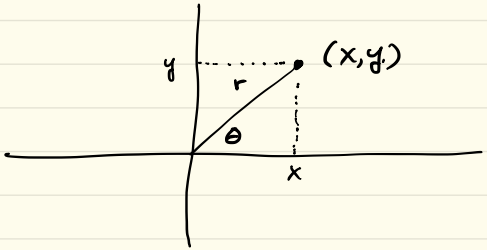
$$1 - (1-p)^{k+1} \geq u$$
$$\Rightarrow \quad 1 - u \geq (1-p)^{k+1}$$
$$\Rightarrow \quad \log_{1-p}(1-u) \geq k+1$$
$$\Rightarrow \quad \frac{\ln(1-u)}{\ln(1-p)} - 1 \geq k$$
$$\Rightarrow \quad \left\lceil \frac{\ln(1-u)}{\ln(1-p)} \right\rceil - 1 = k$$

Normal $N(0,1)$ distribution

inverse transform method difficult because inverting CDF is difficult

TRY A TRANSFORM

Refresher on polar coordinates



A point can be described as (r,θ) in polar coordinates or (x,y) in Cartesian coordinates

relationship $r^2 = x^2 + y^2$ $\tan(\theta) = \frac{y}{x}$

so $x = r\cos(\theta)$

$y = r\sin(\theta)$

Consider joint density function for independent normals

$X \sim N(0,1)$, $Y \sim N(0,1)$

$$p(x,y) = p(x)p(y) = \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)$$

$$= \left(\frac{1}{2\pi}\right)\left(e^{-(x^2+y^2)/2}\right)$$

$$= \left(\frac{1}{2\pi}\right)\left(e^{-r^2/2}\right) \quad \text{where } r^2 = x^2+y^2$$

we can think of $p(x,y)$ as the product of two

different density functions

$\left(\frac{1}{2\pi}\right)$ is density for uniform $(0, 2\pi)$

and $e^{-r^2/2}$?

Consider CDF for r^2 defined by

$$\Pr\{z \le r\} = 1 - e^{-r^2/2}$$, i.e. exponential in r^2

then density function is

$$\frac{d}{dr}\left(1 - e^{-r^2/2}\right) = (2)\left(\tfrac{1}{2}\right) e^{-r^2/2}$$

$$= e^{-r^2/2}$$

$\Rightarrow$ we can view $e^{-r^2/2}$ as the exponential pdf of $X^2 + Y^2 = R^2$

which is $\exp\left(\frac{1}{2}\right)$

remember we sample $\exp(\lambda)$ by $-\frac{1}{\lambda}\ln(1-u)$

$$\sim -\frac{1}{\lambda}\ln(u)$$

so sampling for R^2

$$R^2 = -2\ln(u_1)$$

So we can sample "angle" component by

$$(u_2 \cdot 2\pi)$$

So we're going from sampling (x, y), $x \sim N(0,1)$, $y \sim N(0,1)$

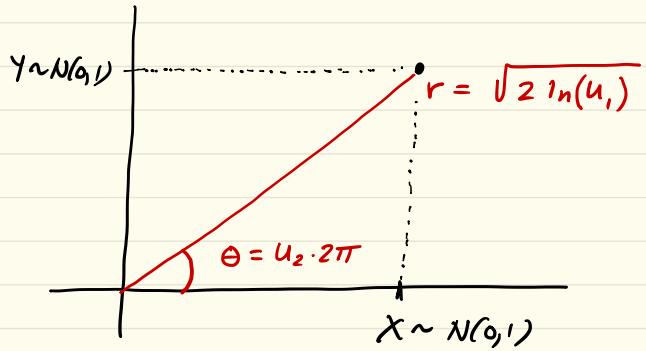
to sampling (θ, R^2)

$$\theta \sim U(0, 2\pi) \qquad R^2 \sim \exp(\tfrac{1}{2})$$

$$\theta = u_2 \cdot 2\pi \qquad R^2 \sim -2\ln(u_1)$$

so

$$R \sim \sqrt{-2 \cdot \ln(u_1)}$$



Now use polar → rectangular coordinate transformation

$$x = r\cos(\theta) = \left(2\ln(u_1)\right)^{1/2} \cdot \cos\left(u_2 \cdot 2\cdot\pi\right)$$

$$y = r\sin(\theta) = \left(2\ln(u_1)\right)^{1/2} \cdot \sin\left(u_2 \cdot 2\cdot\pi\right)$$

So two independent uniform random variables can be used through this transformation to obtain two independent $N(0,1)$ random variables

Transformation of $N(0,1) \rightarrow N(\mu, \sigma)$

Given a $N(0,1)$ normal, it is easy to create a Gaussian with mean μ and standard deviation σ

Given $Z \sim N(0,1)$ and parameters μ, σ, define

$$X = \sigma Z + \mu$$

and X will be Gaussian with mean μ and standard deviation σ

Conversely, if X is Gaussian with mean μ and standard deviation σ

then $Y = \frac{X - \mu}{\sigma}$ is standard normal

These relationships mean that we can infer the CDF values for a Gaussian from the CDF of a standard normal

IF $X \sim N(\mu, \sigma)$ Then

$$F_X(t) = Pr\{X \leq t\}$$

$$= Pr\{\sigma Z + \mu \leq t\}$$

$$= Pr\{Z \leq \frac{t-\mu}{\sigma}\}$$

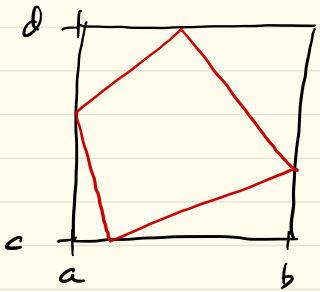
$$= \Phi\left(\frac{t-\mu}{\sigma}\right)$$

REJECTION SAMPLING

BASIC FORM

a) Sample x from an "easy" distribution

b) accept x if it meets some criteria
reject otherwise



example: sample points within red polygon uniformly

1. sample (x, y),
$x \sim U(a, b)$, $y \sim U(c, d)$

2. Use geometry to determine whether (x, y) falls within red polygon

- accept or reject

GENERALIZATION

Let f, g be probability density functions

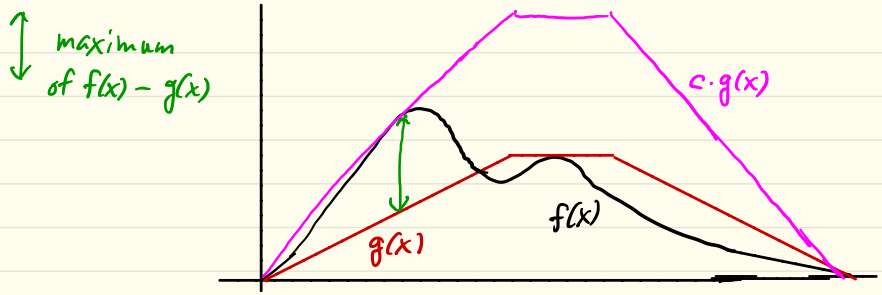
- g is "easy" to sample from
- $f(x)$ can be computed

Suppose there is $c > 1$ such that for all x

$$f(x) \leq c \cdot g(x)$$

Algorithm

1. Sample y from g
2. sample z from $U[0, c \cdot g(y)]$
3. if $z < f(y)$, accept y else reject



$$\Pr\{\text{sample is accepted}\} = \int_{-\infty}^{\infty} g(x) \cdot \frac{f(x)}{c \cdot g(x)} \cdot dx$$

so # samples to accept is geometric $\left(\frac{1}{c}\right)$ $= \frac{1}{c}$

We are interested in the distribution of accepted samples

$$\Pr\{Y \le t \mid Y \text{ is accepted}\} = \frac{\Pr\{Y \le t \ \&\ Y \text{ is accepted}\}}{\Pr\{Y \text{ is accepted}\}}$$

$$= \frac{\int_{x \le t} g(x) \cdot \frac{f(x)}{c \cdot g(x)} \cdot dx}{\frac{1}{c}}$$

$$= \int_{x \le t} f(x) \cdot dx = F(t)$$

∎

Application to Normal

Find $g(x)$, c, such that $\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \le c \cdot g(x)$ for all x

Try $g(x)$ based on exponential
– with probability $1/2$ go positive,

$g(x) = \frac{1}{2} \cdot e^{-x}$ (here $\lambda = 1$, $1/2$ for selection of direction)

want c s.t. $c \cdot g(x) \ge f(x)$ for all x

$$\Rightarrow c \cdot \frac{1}{2} \cdot e^{-x} \ge \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

$$\Rightarrow c \ge \frac{2}{\sqrt{2\pi}} \cdot e^{-(x^2/2 - x)} \quad \text{for all } x$$

Now $-(x^2/2 - x)$ is maximized at $x = 1$ (use calculus)

$$\Rightarrow c \ge \frac{2}{\sqrt{2\pi}} \cdot e^{1/2} = \sqrt{\frac{4 \cdot e}{2\pi}}$$

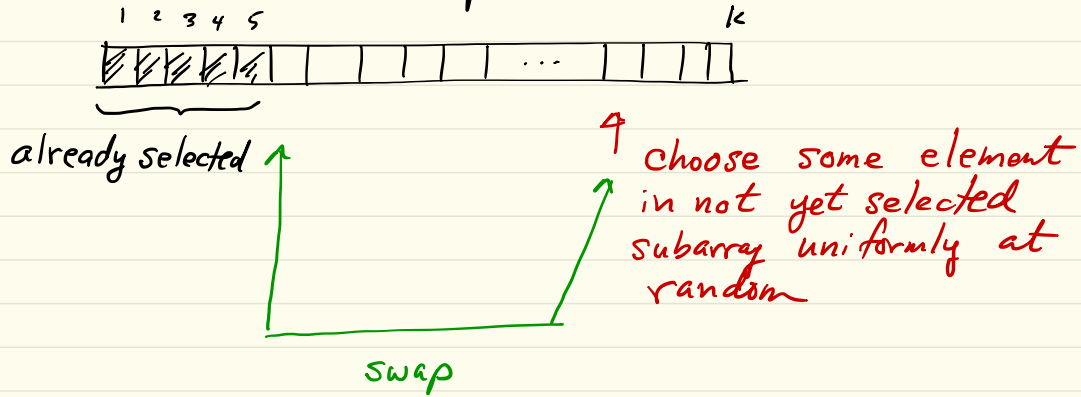
$$= \sqrt{\frac{2 \cdot e}{\pi}} \approx 1.315$$

Creating uniformly random permutation

Problem: Given sequence $d_1, d_2, d_3, \ldots, d_k$ generate a permutation which is exactly as likely as any other

Note: there are $k!$ permutations, so a uniformly random one will have probability $1/k!$

BASIC IDEA: Permute "in place"



```
for i in xrange(0, k-1):
    j = random.randint(i, k-1)
    swp = L[i]
    L[i] = L[j]
    L[j] = swp
```

Proof of correctness:

$$Pr\{S_1, S_2, S_3, \ldots, S_k\}$$

$$= Pr\{\text{select } S_1 \text{ from } k\} \cdot Pr\{\text{select } S_2 \text{ from } k-1\} \cdot \ldots$$

$$= \frac{1}{k} \cdot \frac{1}{k-1} \cdot \frac{1}{k-2} \cdot \ldots \cdot \frac{1}{2}$$

$$= \frac{1}{k!}$$

CONVOLUTION METHODS

- some random variates are naturally expressed as a sum of others

- Sum of N Bernoulli r.v.s is a binomial
- Sum of k exponentials is an Erlang-k

Some computational issues to consider
memory / computation

example: Binomial
- constant (small) space, N samples from $U(0,1)$, $O(N)$ time
- use inverse dist. function
  - $O(N)$ time to create, $O(N)$ space
    $O(\log N)$ time per subsequent sample

- alias table: $O(N)$ space, $O(N)$ time to create, $O(1)$ time to sample

Alias Table — Big Picture

- Works for finite discrete distributions
- Given N possible values, create table with N entries
  EACH entry has form (q_i, V_j, V_k)

  where q_i is a probability, V_j and V_k are possible values

TO SAMPLE

1. CHOOSE i^{th} table entry (q_i, V_j, V_k) uniformly at random
2. WITH probability q_i select V_j as sample, else V_k

THAT'S ALL!

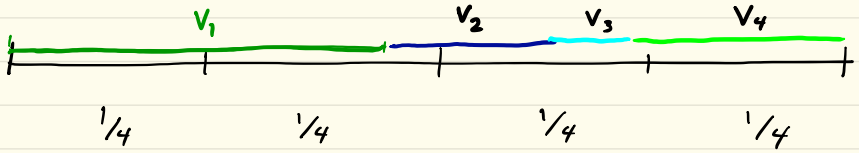
NOTE — a value V_j may appear in multiple entries

— the magic part is in setting up the entries

BASIC IDEA

imagine values laid out on $[0, 1]$ with length proportional to probability

$\Pr\{V_1 = \frac{2}{5}\}$, $\Pr\{V_2 = \frac{1}{5}\}$ $\Pr\{V_3 = \frac{1}{8}\}$ $\Pr\{V_4 = \frac{11}{40}\}$



Key idea is spreading a sample's probability mass among equi-sized bins

Some care needed though to limit at most 2 samples per bin (notice V_2, V_3 and V_4 have mass in same bin above)

CREATING ALIAS TABLE

1. Partition values into two sets
   - G : those with probability $\leq 1/N$
   - H : all others

2. For every $i = 1$ to N
   $r_i = Pr\{V_i\}$

3. For $i = 1$ to $N-1$
   - Choose any V_j in G
   - set $q_i = r_j \cdot N$ // assured to be < 1
   - make V_j first component of i^{th} table entry
   - Choose $V_k \in H$, make 2nd component
   - Set $r_k = r_k - (1 - q_i)/N$ // subtract allocation to i

   Last table entry takes single value left in H - $(1, V_e, -)$

note on reset of r_k: subtract off product of selecting entry ($1/N$) times selecting V_k

EXAMPLE ALIAS TABLE CONSTRUCTION

$Pr\{V_1 = \frac{2}{5}\}$, $Pr\{V_2 = \frac{1}{5}\}$ $Pr\{V_3 = \frac{1}{8}\}$ $Pr\{V_4 = \frac{11}{40}\}$

1. $G = \{V_2, V_3\}$, $H = \{V_1, V_4\}$

$r_1 = 2/5$, $r_2 = 1/5$, $r_3 = 1/8$, $r_4 = 11/40$

2. Table construction

	From G	From H	entry	residual	action
1.	V_2	V_1	$(\frac{4}{5}, V_2, V_1)$	$\frac{2}{5} - \frac{1}{20} = \frac{7}{20}$	$V_1 \rightarrow H$
2.	V_3	V_1	$(\frac{1}{2}, V_3, V_1)$	$\frac{7}{20} - \frac{1}{8} = \frac{9}{40}$	$V_1 \rightarrow G$
3.	V_1	V_4	$(\frac{9}{10}, V_1, V_4)$	$\frac{11}{40} - \frac{1}{40} = \frac{10}{40}$	$V_4 \rightarrow G$
4.	V_4	–	$(1.0, V_4, -)$		