Gambler's Ruin

$P_{i,i+1} = p_{up}$, $P_{i,i-1} = 1 - p_{up}$ for $i = 1, 2, \ldots, N-1$

$P_{0,0} = 1$, $P_{N,N} = 1$

Ehrenfest Urn Model

2 urns, N balls

Each step choose a ball at random, change its urn

state is # balls in Urn,

$$P_{i,i-1} = P(\text{Urn, ball selected}) = \frac{i}{N}$$

$$P_{i,i+1} = 1 - P_{i,i+1}$$

insight into gas dynamics

Branching Process

$X(n)$ = number of individuals in n^{th} generation
$X(0) = 1$

Each generation, each individual either

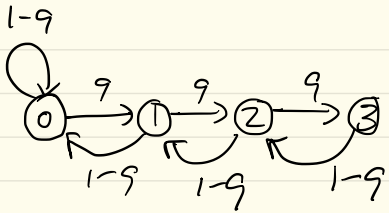
- dies, or
- has a random number of offspring

So each generation, each individual is replace by random Y number of individuals

$$X(n+1) = \sum_{i=1}^{X(n)} Y_i \qquad \text{where } Y_i \text{ are i.i.d}$$

$$P_{i,j} = P\left(\sum_{k=1}^{i} Y_k = j\right)$$

Simple random walk
- on integers
- reflection at 0

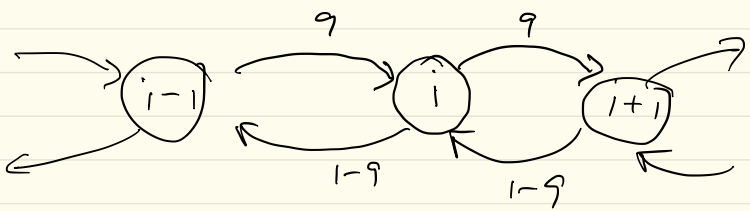


$p_{i,i+1} = q$ for all $i = 0, 1, \ldots$

$p_{0,0} = 1 - q$

Solve for π_i

Flow balance equations



$\pi_i (q + (1-q)) =$

$q \cdot \pi_{i-1} + (1-q) \cdot \pi_{i+1}$

OUT IN

$\pi_i (1-q+q) = \pi_{i-1} \cdot q + \pi_{i+1} \cdot (1-q)$

State 0 OUT IN

$$\pi_0 = (1-q)\pi_0 + (1-q)\pi_1$$

$$1 - (1-q)\pi_0 = (1-q)\pi_1$$

$$q\pi_0 = (1-q)\pi_1$$

$$\left(\frac{q}{1-q}\right)\pi_0 = \pi_1$$

State 1

$$\pi_1 = q\cdot\pi_0 + (1-q)\pi_2$$

$$\downarrow$$

So $$\left(\frac{q}{1-q}\right)\pi_0 = q\cdot\pi_0 + (1-q)\pi_2$$

$$\left(\frac{q}{1-q}\right)\pi_0 = q\cdot\pi_0 + (1-q)\pi_2$$

$$\left(\frac{q}{1-q} - \frac{q(1-q)}{1-q}\right)\pi_0 = (1-q)\pi_2$$

$$\left(\frac{q^2}{1-q}\right)\pi_0 = (1-q)\pi_2$$

$$\left(\frac{q}{1-q}\right)^2\cdot\pi_0 = \pi_2$$

$$\vdots$$

$$\left(\frac{q}{1-q}\right)^k\cdot\pi_0 = \pi_k$$

how to solve?

$$\sum_{k=0}^{\infty}\left(\frac{q}{1-q}\right)^k\cdot\pi_0 = 1$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(\frac{q}{1-q}\right)^k = \frac{1}{\pi_0}$$

recall when $\alpha < 1$

$$\sum_{k=0}^{\infty}\alpha^k = \frac{1}{\alpha}$$

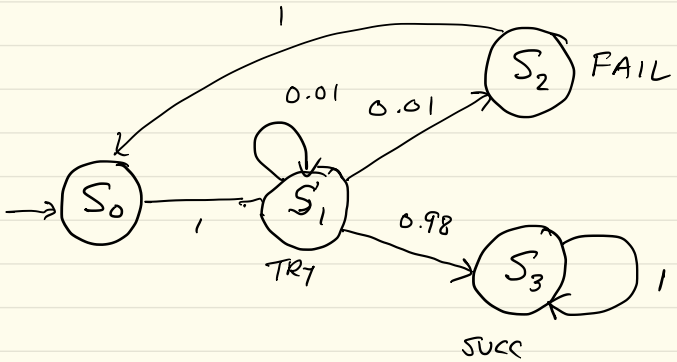
so when $\frac{q}{1-q} < 1$

$$\frac{q}{1-q} = \pi_0$$

and $$\pi_k = \left(\frac{q}{1-q}\right)^{k+1}$$

Simple Protocol

- after 1 step process starts comm. attempt
  - channel busy with prob. 0.01
  - message succeeds with prob 0.98
  - message fails, retry with prob 0.01



Question – dist. of time slots required to succeed

Notice

- S_3 is an "absorbing" state

$\pi^{(n)}(S_3)$ = P(chain enters S_3 by time n or earlier)

So $\pi^{(n)}(S_3)$ for $n = 1, 2, \ldots, \infty$

is CDF of time to send message

ASIDE

$$E[X] = \sum_{n=1}^{\infty} P\{X \geq n\} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\{X = k\}$$

$$= \sum_{j=1}^{\infty} j \cdot Pr\{X = j\}$$

Availability Example

- system may fail, w.p. p_f @ day
  - can be brought back instantly if spare is available

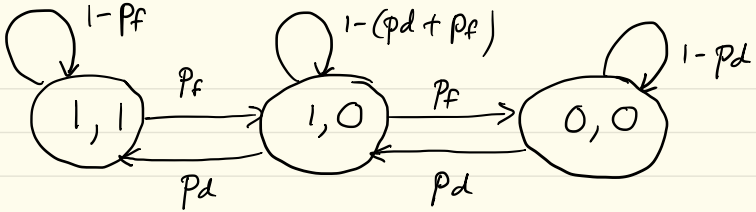
State = (b_1, b_2) b_1 : Boolean = 1 if system up

b_2 : Boolean = 1 if spare present

p_f : prob of failure

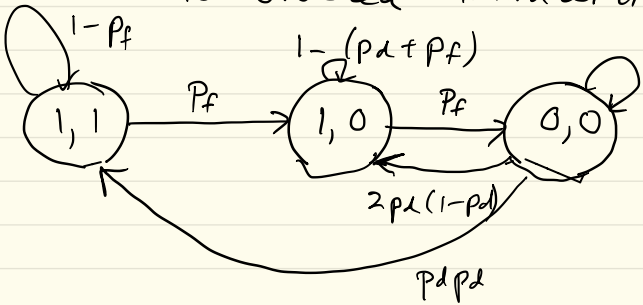
p_d : prob ordered spare is delivered

- assume that spare is ordered when spare placed into service



- what is availability?

Now suppose that when system fails another spare is ordered immediately

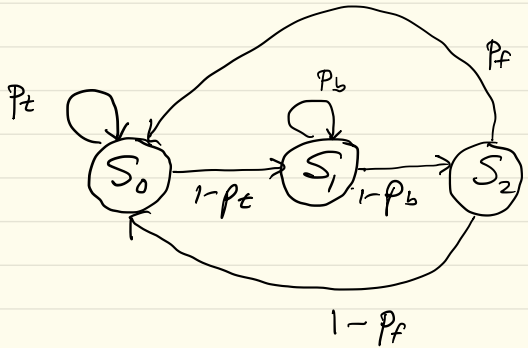


$1 - (p_d p_d + 2p_d(1-p_d))$

$= 1 - 2p_d + p_d p_d$

Revisit simple protocol

- in state S_0 "think" for a geometrically distributed time with success parameter p_t
- in state S_1 channel is busy with prob p_b
  message fails with probability p_f, returns to S_0
  successful message returns to S_0 w.p. $1-p_f$



What is the thruput?

$\Pi_2 \cdot (1-p_f)$

why?

(S, N) inventory system

- maximum N items in stock
- inventory replenished at end of day in $\leq s$
- D is distribution of demand @ day
  i.i.d.

state $\equiv$ items in inventory at beginning of day

State is N at beginning of day if either it started previous day and had demand pushing below $s+1$, or was at

$$p_{N,N} = P\{D=0\}$$

$$p_{j,N} = P\{j - D \leq s\} = P\{j - s \leq D\}$$

$k < N$

$$p_{j,k} = P\{j - D = k\}$$

Questions

- what is the probability of unfulfilled demand?

Suppose

* cost to retailer of purchasing one item is c_p
* cost of holding 1 item in inventory is c_I per day
* sale price per item is c_s

what values of s, N maximize profit?