ECE 534 SP26 HW7 Solutions

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We only provide the solution to [1, Problem 3.9]. The solution to [1, Problem 4.7] can be found in the solution set for HW5, and the solutions to the other problems can be seen in [1].

Solution. (Solution to [1, Problem 3.9].)

(a) The best estimator of X of the form $g(U)$ is given by $g(u) = \frac{1}{1+u}$, because this estimator has MSE equal to zero. That is, $E[X|U] = X$.

(b) We need to calculate some moments:

$$E[U] = \int_0^1 u\,du = \frac{1}{2}.$$

$$E[U^2] = \int_0^1 u^2\,du = \frac{1}{3}.$$

$$\text{Var}(U) = E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

$$E[X] = \int_0^1 \frac{1}{1+u}\,du = \ln(1+u)\Big|_0^1 = \ln 2.$$

$$E[X^2] = \int_0^1 \frac{1}{(1+u)^2}\,du = -\frac{1}{1+u}\Big|_0^1 = \frac{1}{2}.$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{1}{2} - (\ln 2)^2.$$

$$E[XU] = \int_0^1 \frac{u}{1+u}\,du = 1 - \int_0^1 \frac{1}{1+u}\,du = 1 - \ln 2.$$

$$\text{Cov}(X,U) = E[XU] - E[X]E[U] = (1-\ln 2) - (\ln 2)\left(\frac{1}{2}\right) = 1 - \frac{3}{2}\ln 2.$$

So we have

$$\begin{aligned}\widehat{E}[X|U] &= E[X] + \frac{\text{Cov}(X,U)}{\text{Var}(U)}(U - E[U]) \\ &= \ln 2 + \frac{1 - \frac{3}{2}\ln 2}{\frac{1}{12}}\left(U - \frac{1}{2}\right) \\ &= \ln 2 + (12 - 18\ln 2)\left(U - \frac{1}{2}\right),\end{aligned}$$

and the mean square error is

$$\begin{aligned}\text{Var}(X) - \frac{\text{Cov}(X,U)^2}{\text{Var}(U)} &= \left(\frac{1}{2} - (\ln 2)^2\right) - \frac{\left(1 - \frac{3}{2}\ln 2\right)^2}{\frac{1}{12}} \\ &= -\frac{23}{2} + 36\ln 2 - 28(\ln 2)^2.\end{aligned}$$

References

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf