University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 534 Random Processes

Spring 2005

Problem Set 6

Ergodicity, Karhunen-Loeve Expansions, Random Processes through Linear Systems, Wiener Filtering

Issued: Wednesday, April 20th **Due:** Beginning of lecture on Wednesday, May 4

Reading from Hajek: Chapter 5, Sections 5.4–5.6; Chapter 6 (in particular, Sections 6.1–6.3); Chapter 7 (in particular, Sections 7.1–7.4).

Reading from Stark and Woods: Chapter 8, Section 8.4–8.5; Chapter 7, Sections 7.3–7.5; Chapter 9, Section 9.3.

Announcement: The Final Exam is scheduled for Thursday, May 12th, from 1:30-4:30pm in room 1214 Siebel. The exam is closed-book but you can bring *three* 8.5 × 11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary. The final exam covers everything from the beginning of the course up to (and including) the lecture on Wednesday, May 4th and up to (and including) Problem Set 6.

Problem 6.1

This problem concerns ergodicity for random processes.

(a) Let $X(t)$ be a wide-sense stationary Gaussian random process with zero-mean and correlation function

$$R_X(\tau) = \sigma^2 e^{-\alpha|\tau|} \cos(2\pi f \tau) ,$$

where σ^2, α and f are all constants. Show that $X(t)$ is ergodic in the mean.

(b) Assume that a wide-sense stationary random process $X(t)$ is ergodic in the mean and that the limit of $K_X(\tau)$ as $\tau \to \infty$ exists. Show that

$$\lim_{|\tau|\to\infty} K_X(\tau) = 0 .$$

Problem 6.2

From Hajek, Chapter 5: Problems 5 and 8.

Problem 6.3 (optional)

From Hajek, Chapter 5: Problems 7 and 9.

Problem 6.4

Let N_t, $t \geq 0$, be a Poisson random process with rate $\lambda > 0$.

(a) In which sense is N_t continuous? Almost surely? In the mean-square sense? In probability? In distribution?

(b) Let $T > 0$ and describe the Karhunen-Loeve expansion for the centered process $N_t - \lambda t$ in the interval $[0, T]$ (i.e., provide the basis functions and the corresponding eigenvalues).

Problem 6.5

From Hajek, Chapter 6: Problems 1 and 2.

Problem 6.6

From Hajek, Chapter 6: Problems 3 and 4.

Problem 6.7

Let X_t be an arbitrary stochastic process and $\widehat{X}_t$ be the MMSE optimal linear estimate of the current value of X_t based on just two past values X_{t_1} and X_{t_2} for $t_1 < t_2 < t$. What are the necessary and sufficient conditions on $R_X(t, s)$ such that $\widehat{X}_t$ only depends on the most recent value X_{t_2}?

Problem 6.8

From Hajek, Chapter 7: Problems 1 and 3.

Problem 6.9 (optional)

From Hajek, Chapter 7: Problem 2.

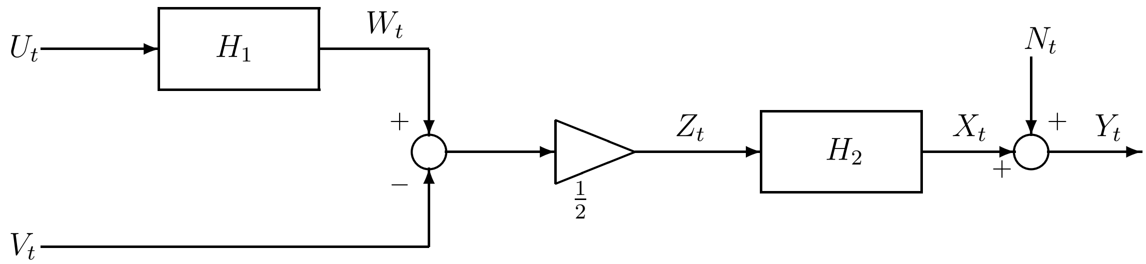
Problem 6.10

You are given three wide-sense stationary (WSS) random processes U_t, V_t and N_t which are zero mean, mutually uncorrelated and have power spectral densities as given below

$$S_U(\omega) = 1\,, \quad S_V(\omega) = 4\,, \quad S_N(\omega) = \frac{3}{1+\omega^2}\,.$$

We introduce four other random processes, Y_t, X_t, Z_t and W_t, related to U_t, V_t and N_t via the block diagram below where

$$H_1(\omega) = \frac{2}{1+j\omega}\,, \quad H_2(\omega) = \frac{1}{2+j\omega}\,.$$



(a) Obtain expressions for the individual power spectral densities of Y_t and X_t, and the cross spectral density between Y_t and X_t.

(b) Determine the filter transfer function $H_3(\omega)$ such that its output $\widehat{X}_t$ provides a linear MMSE estimate for X_t based on the output process Y_t.

Problem 6.11

Let X_t be a real-valued, zero-mean stationary Gaussian process with $R_X(\tau) = e^{-|\tau|}$. Let $\alpha > 0$ and suppose that X_0 is estimated by $\widehat{X}_0 = c_1 X_{-\alpha} + c_2 X_\alpha$, where the constants c_1 and c_2 are chosen to minimize the mean square error (MSE).

(a) Find the optimal choice for c_1 and c_2 and the corresponding MSE $E[(X_0 - \widehat{X}_0)^2]$.

(b) Use the orthogonality principle to show that $\widehat{X}_0$ as defined above is the MMSE estimator of X_0 given X_s for $|s| \geq \alpha$.

Problem 6.12

From Hajek, Chapter 7: Problems 4 and 5.

Problem 6.13 (optional)

From Hajek, Chapter 7: Problems 6 and 7.