University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 534 Random Processes

Spring 2005

Problem Set 5

Markov Random Processes, Calculus of Random Processes, Ergodicity

Issued: Wednesday, March 30th **Due:** Never (covered in Midsemester Exam 2)

Reading from Hajek: Chapter 4; Sections 4.8–4.9; Chapter 5, Sections 5.1–5.4.

Reading from Stark and Woods: Chapter 7, Sections 7.1–7.2; Chapter 8, Section 8.1–8.4.

Announcement: Midsemester Exam 2 is scheduled for Wednesday, April 13th, from 5:00-7:00pm in room DCL1310. The exam is closed-book but you can bring *two* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary. The exams covers everything from the beginning of the course up to (and including) the lecture on Monday, April 11th and up to (and including) Problem Set 5.

Problem 5.1

From Hajek, Chapter 4: Problems 8 and 9.

Problem 5.2

A random telegraph signal is a random process $X(t)$ defined for $t \geq 0$ as follows: $X(0) = +1$ and $X(t)$ switches between $+1$ and -1 according to a Poisson random arrival sequence $T[n]$, i.e.

$$X(t) = \begin{cases} 1, & 0 \leq t < T[1], \\ -1, & T[1] \leq t < T[2], \\ 1, & T[2] \leq t < T[3], \\ \cdots & \cdots \end{cases}$$

Assume that the rate parameter λ of the Poisson random arrival time sequence is known.

(a) Argue that $X(t)$ is a Markov process and draw and label its state transition diagram.

(b) Find the steady-state probability that $X(t) = +1$ in terms of the rate parameter λ.

Problem 5.3 (optional)

From Hajek, Chapter 4: Problem 11.

Problem 5.4

From Hajek, Chapter 5: Problems 1, 2, 3 and 4.

Problem 5.5

Let the random process $X(t)$ be a stationary random process with mean μ_X and covariance function

$$K_{XX}(\tau) = \frac{\sigma_X^2}{1 + \alpha^2 \tau^2} .$$

(a) Show that a mean-square derivative exists for all t.

(b) Find $\mu_{X'}(t)$ and $K_{X'X'}(\tau)$ for all t and τ.

Problem 5.6

Let $X(t)$ be a wide-sense stationary Gaussian random process. Show that the m.s. derivative of

$$Y(t) \equiv X^2(t)$$

is $Y'(t) = 2X(t)X'(t)$ and find the correlation function of $Y'(t)$ in terms of $R_X(\tau)$ and its derivatives. [**Hint:** Recall that for jointly Gaussian random variables $E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$.]

Problem 5.7

From Hajek, Chapter 5: Problems 5 and 6.

Problem 5.8

This problem concerns ergodicity for random processes.

(a) State the general definition of "ergodic in the mean" for a wide-sense stationary process $X(t)$.

(b) Let $X(t)$ be a wide-sense stationary Gaussian random process with zero-mean and correlation function

$$R_X(\tau) = \sigma^2 e^{-\alpha|\tau|} \cos(2\pi f \tau) ,$$

where σ^2, α and f are all constants. Show that $X(t)$ is ergodic in the mean.

(c) Assume that a wide-sense stationary random process $X(t)$ is ergodic in the mean and that the limit of $K_X(\tau)$ as $\tau \to \infty$ exists. Show that

$$\lim_{|\tau| \to \infty} K_X(\tau) = 0 .$$