University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 534: Random Processes

Spring 2005

Problem Set 3

Random Vectors, Minimum Mean Squared Error Estimation, Discrete-Time Kalman Filtering

Issued: Wednesday, March 2nd **Due:** Never (covered in Midsemester Exam 1)

Reading from Hajek: Chapter 3.

Reading from Stark and Woods: Chapter 4, Section 4.8; Chapter 5.

Announcement: Midsemester Exam 1 is scheduled for Wednesday, March 9th, from 5:00-7:00pm in room DCL1310. The exam is closed-book but you can bring an 8.5×11-inch double-sided sheet of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 3.1 (optional)

Let X and Y be two random variables with zero means, unit variances and correlation coefficient ρ. Show that

$$E[\max(X,Y)] \leq 1 + \sqrt{1-\rho^2}\ .$$

Problem 3.2

From Hajek, Chapter 3: Problem 2.

Problem 3.3

Let random variables X and Y have joint density

$$f(x,y) = \begin{cases} x+y, & \text{if } x,y \in [0,1], \\ 0, & \text{else.} \end{cases}$$

(a) Find the *linear* MMSE estimator of X given Y.

(b) What is the mean squared error of the LMMSE you computed in part (a)?

(c) Find the (possibly nonlinear) MMSE estimator of X given Y.

(d) What is the mean squared error of the MMSE you computed in part (c)?

(e) Compute the ratio of the mean squared errors, i.e., divide your answer in part (d) (the MMSE squared-error) by your answer in part (b) (the LMMSE squared error).

Problem 3.4

Repeat parts (a) through (e) of the previous problem for the case when Y is a $N(0,1)$ random variable and $X = |Y|$.

Problem 3.5

Let Θ be a uniformly distributed random variable on the interval $[0, \pi]$ (not $[0, 2\pi]$). Suppose random variable $Y = \cos(\Theta)$ is to be estimated via a linear estimator of the form $\widehat{Y}_{LMMSE}(\theta) = a + b\theta$. Find constants a and b such that the mean squared error is minimized.

Problem 3.6

Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a Gaussian random vector with mean and covariance matrix as given below

$$\mu_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad K_X = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $-1 \le \rho \le 1$. Find an orthonormal 2×2 matrix U such that $X = UY$ for a Gaussian vector $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ such that Y_1 and Y_2 are independent. What are the distributions of Y_1 and Y_2?

Problem 3.7

From Hajek, Chapter 3: Problems 4, 5 and 6.

Problem 3.8

Show that for jointly Gaussian random vectors $\vec{X}$ and $\vec{Y}$, the *linear* MMSE estimate of $\vec{X}$ and the (possibly non-linear) MMSE estimate of $\vec{X}$ based on observations $\vec{y}$ are identical.

Problem 3.9

Consider a 3-dimensional Gaussian random vector $\vec{x}$ with zero mean and covariance matrix

$$\Sigma = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Find a matrix A such that $\vec{x} = A\vec{z}$, where $\vec{z}$ is another 3-dimensional Gaussian random vector, with zero mean and covariance the *identity* matrix.

(b) Determine $E[X_1 | X_2, X_3]$, where X_i is the i-th component of $\vec{x}$.

Problem 3.10

Consider the following communication problem, in which a length-n vector $\vec{x}$ of bits are to be transmitted over a noisy channel to a receiver. The relationship between the transmitted data $\vec{x}$ and the received length-m vector $\vec{y}$ is given by $\vec{y} = H\vec{x} + \vec{\omega}$, where H is a known $m \times n$ matrix representing the effects of the LTI channel (i.e., it is a convolution matrix), and $\vec{\omega}$ is $N(0, \sigma^2 I)$. Note that n need not equal m due to edge effects from the channel. The data is transmitted using BPSK, i.e., X_1, X_2, ..., X_n are i.i.d. and $P(x_i = 1) = P(x_i = -1) = \frac{1}{2}$ for $i = 1, ..., n$.

(a) Determine the joint density of $\vec{x}$ and $\vec{y}$.

(b) For an estimator of the form $\hat{\vec{x}}(\vec{y}) = A\vec{y} + \vec{b}$, determine the values of matrix A and vector $\vec{b}$, such that the error $\epsilon = E[(\vec{x} - \hat{\vec{x}}(\vec{y}))^2]$ is minimized.

(c) If the estimator $\hat{\vec{x}}(\vec{y})$ is not restricted to be of the form given in part (b), determine $\hat{\vec{x}}(\vec{y})$ that minimizes the error $\epsilon = E[(\vec{x} - \hat{\vec{x}}(\vec{y}))^2]$. For this part of the problem (and this part only) assume that x, y and H are scalars, i.e., $m = n = 1$.

(d) Determine an upper bound on ϵ from part (c).

Problem 3.11

Let the random vector $[X\,Y\,Z]^T$ be a Gaussian random vector with mean $[1\;2\;3]^T$ and covariance matrix

$$\Sigma = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) Suppose that X is estimated by an estimator of the form $\hat{X} = aY^2 + bY + c$. Determine numerical values of a, b and c such that $\epsilon = E((X - \hat{X})^2)$ is minimized.

(b) Determine the resulting numerical value of the mean squared error from part (a).

(c) Suppose that X is estimated by an estimator of the form $\hat{X} = aY^2$. Determine the numerical value of a such that $\epsilon = E((X - \hat{X})^2)$ is minimized. (**Hint:** The moment generating function of a Gaussian vector x with distribution $N(m, K)$ is $\Phi(u) = exp(-\frac{1}{2}u^T K u + ju^T m)$.)

Problem 3.12

From Hajek, Chapter 3: Problem 7.