

Problem Set 3

Random Vectors, Minimum Mean Squared Error Estimation, Discrete-Time Kalman Filtering

Issued: Wednesday, March 2nd

Due: Never (covered in Midsemester Exam 1)

Reading from Hajek: Chapter 3.

Reading from Stark and Woods: Chapter 4, Section 4.8; Chapter 5.

Announcement: Midsemester Exam 1 is scheduled for Wednesday, March 9th, from 5:00-7:00pm in room DCL1310. The exam is closed-book but you can bring an 8.5×11 -inch double-sided sheet of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 3.1 (optional)

Let X and Y be two random variables with zero means, unit variances and correlation coefficient ρ . Show that

$$E[\max(X, Y)] \leq 1 + \sqrt{1 - \rho^2}.$$

Problem 3.2

From Hajek, Chapter 3: Problem 2.

Problem 3.3

Let random variables X and Y have joint density

$$f(x, y) = \begin{cases} x + y, & \text{if } x, y \in [0, 1], \\ 0, & \text{else.} \end{cases}$$

- (a) Find the *linear* MMSE estimator of X given Y .
- (b) What is the mean squared error of the LMMSE you computed in part (a)?
- (c) Find the (possibly nonlinear) MMSE estimator of X given Y .
- (d) What is the mean squared error of the MMSE you computed in part (c)?
- (e) Compute the ratio of the mean squared errors, i.e., divide your answer in part (d) (the MMSE squared-error) by your answer in part (b) (the LMMSE squared error).

Problem 3.4

Repeat parts (a) through (e) of the previous problem for the case when Y is a $N(0, 1)$ random variable and $X = |Y|$.

Problem 3.5

Let Θ be a uniformly distributed random variable on the interval $[0, \pi]$ (not $[0, 2\pi]$). Suppose random variable $Y = \cos(\Theta)$ is to be estimated via a linear estimator of the form $\hat{Y}_{LMMSE}(\theta) = a + b\theta$. Find constants a and b such that the mean squared error is minimized.

Problem 3.6

Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a Gaussian random vector with mean and covariance matrix as given below

$$\mu_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad K_X = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $-1 \leq \rho \leq 1$. Find an orthonormal 2×2 matrix U such that $X = UY$ for a Gaussian vector $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ such that Y_1 and Y_2 are independent. What are the distributions of Y_1 and Y_2 ?

Problem 3.7

From Hajek, Chapter 3: Problems 4, 5 and 6.

Problem 3.8

Show that for jointly Gaussian random vectors \vec{X} and \vec{Y} , the *linear* MMSE estimate of \vec{X} and the (possibly non-linear) MMSE estimate of \vec{X} based on observations \vec{y} are identical.

Problem 3.9

Consider a 3-dimensional Gaussian random vector \vec{x} with zero mean and covariance matrix

$$\Sigma = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

- (a) Find a matrix A such that $\vec{x} = A\vec{z}$, where \vec{z} is another 3-dimensional Gaussian random vector, with zero mean and covariance the *identity* matrix.
- (b) Determine $E[X_1|X_2, X_3]$, where X_i is the i -th component of \vec{x} .

Problem 3.10

Consider the following communication problem, in which a length- n vector \vec{x} of bits are to be transmitted over a noisy channel to a receiver. The relationship between the transmitted data \vec{x} and the received length- m vector \vec{y} is given by $\vec{y} = H\vec{x} + \vec{\omega}$, where H is a known $m \times n$ matrix representing the effects of the LTI channel (i.e., it is a convolution matrix), and $\vec{\omega}$ is $N(0, \sigma^2 I)$. Note that n need not equal m due to edge effects from the channel. The data is transmitted using BPSK, i.e., X_1, X_2, \dots, X_n are i.i.d. and $P(x_i = 1) = P(x_i = -1) = \frac{1}{2}$ for $i = 1, \dots, n$.

- (a) Determine the joint density of \vec{x} and \vec{y} .
- (b) For an estimator of the form $\hat{\vec{x}}(\vec{y}) = A\vec{y} + \vec{b}$, determine the values of matrix A and vector \vec{b} , such that the error $\epsilon = E[(\vec{x} - \hat{\vec{x}}(\vec{y}))^2]$ is minimized.
- (c) If the estimator $\hat{\vec{x}}(\vec{y})$ is not restricted to be of the form given in part (b), determine $\hat{\vec{x}}(\vec{y})$ that minimizes the error $\epsilon = E[(\vec{x} - \hat{\vec{x}}(\vec{y}))^2]$. For this part of the problem (and this part only) assume that x, y and H are scalars, i.e., $m = n = 1$.
- (d) Determine an upper bound on ϵ from part (c).

Problem 3.11

Let the random vector $[X \ Y \ Z]^T$ be a Gaussian random vector with mean $[1 \ 2 \ 3]^T$ and covariance matrix

$$\Sigma = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

- (a) Suppose that X is estimated by an estimator of the form $\hat{X} = aY^2 + bY + c$. Determine numerical values of a, b and c such that $\epsilon = E((X - \hat{X})^2)$ is minimized.
- (b) Determine the resulting numerical value of the mean squared error from part (a).
- (c) Suppose that X is estimated by an estimator of the form $\hat{X} = aY^2$. Determine the numerical value of a such that $\epsilon = E((X - \hat{X})^2)$ is minimized. (**Hint:** The moment generating function of a Gaussian vector x with distribution $N(m, K)$ is $\Phi(u) = \exp(-\frac{1}{2}u^T K u + ju^T m)$.)

Problem 3.12

From Hajek, Chapter 3: Problem 7.