University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 534: Random Processes

Spring 2005

Problem Set 1

Probability Review

Issued: Wednesday, Jan. 26th **Due:** Beginning of lecture on Monday, Feb. 7

Reading from Hajek: Chapter 1.

Reading from Stark and Woods: Chapters 1, 2 and 3.

Most of this material should be familiar to you from previous courses.

Announcement: The Probability Quiz is scheduled for Wednesday, February 9th, from 5:00-7:00pm in room DCL1310. The quiz is closed-book; calculators are allowed but will not be necessary.

Problem 1.1

From Hajek, Chapter 1: Problems 1, 2, 3, 5, 6, 7 and 9.

Problem 1.2

A man has two coins in his pockets: one is a fair coin with a head and a tail, the other is a special coin with two heads. The man picks out a coin randomly, tosses it and observes if he gets a head or a tail. After that, he puts the coin back to the pocket and repeats the procedure for one more time. Specify the set of possible outcomes, i.e., the sample space Ω; the event space, i.e., the σ-field F; the probability measure P on each event. What is the probability that a head was obtained in the first toss given that the second toss is a tail?

Problem 1.3

Consider a probability space (Ω, F, P).

(a) Under what conditions are $A, A^c \in F$ independent?

(b) If $C \in F$ and $P(C) > 0$, under what conditions are A, A^c conditionally independent given C?

Problem 1.4

For a probability space (Ω, F, P), prove that for all $B \in F$, if $P(B) > 0$, then the conditional probability $P(\cdot|B)$ is a probability measure defined on F.

Problem 1.5

Let

$$P[[a,b)] = \begin{cases} b-a & \text{for } 0 \le a < b \le 1/2\ , \\ \frac{2}{3}(b^2-a^2) & \text{for } 1/2 \le a < b \le 1\ . \end{cases}$$

What must $P[[a,b)]$ be in the range $a \le 1/2 < b$ in order for P to satisfy the axioms of a probability measure? Can P be put in the form $P[[a,b)] = \int_a^b f(x)dx$ for some appropriate $f(x)$? If so, what is $f(x)$?

Problem 1.6

Which of the following are valid pdf's for a continuous random variable X? If the pdf is valid, find the expected value of X; if not, explain why.

(a) $f(x) = \exp(\pi(x-1))$, $-\infty < x < +\infty$;

(b) $f(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & -1.5 < x < 1.5\ , \\ 0, & \text{elsewhere}\ ; \end{cases}$

(c) $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < +\infty$.

Problem 1.7

Let X, Y and Z be independent and identically distributed (i.i.d.) *nonnegative* random variables with density $f(\alpha) = e^{-\alpha}$ for $\alpha \ge 0$.

(a) Find the probability density of X, conditioned on the event that $X \le 1$.

(b) Find the probability density of X, conditioned on the event that $X \le 1$ and $X + Y \le 1$.

(c) Find the joint density of X and Y, conditioned on the event that $X + Y \le 1$ and $X + Y + Z \ge 1$.

(d) Show that the probability density of X conditioned on the event $X + Y \le 1$ and $X + Y + Z \ge 1$ is the same as the probability density of the random variable $\min(U_1, U_2)$ where U_1 and U_2 are two i.i.d. random variables uniformly distributed on $[0, 1]$.

Problem 1.8

The joint pdf of two random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} A(1-|x-y|), & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find A.

(b) Find the marginal pdf of X and Y.

(c) Find $P(X + Y < 1 \mid X > \frac{1}{2})$.

Problem 1.9

Suppose that X and Y are independent Cauchy random variables with densities

$$f_X(x) = \frac{a}{\pi(a^2 + x^2)}\ , \quad f_Y(y) = \frac{b}{\pi(b^2 + y^2)}\ .$$

(a) Show that the characteristic function of X is $\Phi_X(\omega) = e^{-\alpha|\omega|}$.

(b) Use characteristic functions to find the density of $Z = X + Y$.

Problem 1.10 (optional)

From Hajek, Chapter 1: Problem 11.

Problem 1.11 (optional)

This problem serves as an example to many things we have mentioned in class: (a) Not every subset of $[0, 1)$ is necessarily in the Borel σ-field; (b) In general, we cannot allow a σ-field to be closed under the union (hence intersection) of an uncountable number of sets. [To understand the problem you may need to know that the set of rational numbers is countable but the set of real numbers is not countable.]

Consider the interval $[0, 1)$ and its Borel measure P defined as $P[[a, b)] = b - a$ (and extended for countable unions and intersections of such disjoint sets). The steps below lead you to the construction of sets that are not measurable (hence they are not Borel sets).

(a) Introduce the following relation to numbers in $[0, 1)$: say that x and y are related if and only if $x - y$ is a rational number (i.e., $x \sim y$ iff $x - y \in Q$). Prove that $\sim$ is an equivalence relationship that forms a partition of $[0, 1)$ (i.e., show that related numbers belong in the same class and that different classes are disjoint and they cover all numbers in $[0, 1)$).

(b) Construct a set S by choosing (one and only one) element from each of the equivalence classes. By construction, the following is true: $\forall x, y \in S$, $x - y$ must be irrational; therefore, since the set of real numbers is not countable, there must be uncountably many elements in S. For every rational number $q \in [0, 1)$, we may construct a new set $S_q = \{x + q \pmod 1 \mid x \in S\}$ as a subset of $[0, 1)$. Prove that, for two rational numbers $p \neq q$ in $[0, 1)$, the sets S_p and S_q are disjoint.

(c) Prove that every number $x \in [0, 1)$ must be in one of the S_q for some rational number $q \in [0, 1)$. Let I be the set of rational numbers in $[0, 1)$; combining (b) and (c) we have that $\{S_i\}_{i \in I}$ is a countable sequence of disjoint sets and $\cup_{i \in I} S_i = [0, 1)$.

(d) Use the fact that the Borel measure is translation invariant to prove that every so defined S_i is not measurable.