University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 434: Random Processes

Spring 2004

Midsemester Exam 1

Wednesday, March 17, 5:00–7:00pm, 165 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!

- This is a **closed-book** exam! You are allowed one sheet of *handwritten* notes (both sides). Calculators should not be necessary, but feel free to use one.
- **Write your name on the answer booklet.**
- There are **five unequally weighted** problems for a total of **50 points**. A **bonus** problem worth **5 points** is also included. Problems are *not* necessarily in order of difficulty.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may loose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Problem 1 (12/50, equally weighted parts)

This problem has **six independent** true/false questions.

(a) If X and Y are zero-mean random variables, then the linear minimum mean squared error (LMMSE) estimator of X based on the observation $Y = y$ is of the form

$$\widehat{X}(y) = \alpha y$$

for some appropriate constant α.

(b) Random variable X is Gaussian and so is random variable Y. Then, X and Y are independent if they are uncorrelated.

(c) Let X be a random variable in the interval $[-1, 1]$; its cdf is unknown but we know that $E[X^2] = \frac{1}{8}$. Then, it is possible that $P(X \geq \frac{1}{2}) = \frac{3}{4}$.

(d) Random variable X is to be estimated based on observations of the random variable Y via a nonlinear estimator of the form $\widehat{X}(y) = \alpha y^2$. The constant α is chosen so that $E[(X - \alpha Y^2)^2]$ is minimized; then, the following equality is necessarily true

$$E[XY^2] = \alpha E[Y^4] \, .$$

(e) Let X be a random variable with a known distribution that is symmetric about $x = 0$. Given the observation that the random variable $Y = X^2$ takes the value y, the minimum mean squared error (MMSE) estimator $\widehat{X}_{MMSE}(y)$ results in mean squared error $E[(X - \widehat{X}_{MMSE}(Y))^2] = \text{cov}(X)$.

(f) Let $X_1, X_2, ...$ be a sequence of independent (but not identical) random variables with $E[X_i] = 0$ and $\text{cov}(X_i) = 1$ for all i. Then, the sequence of random variables

$$Y_n = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$$

converges in distribution to a Gaussian random variable X with zero mean and unit variance.

Problem 2 (10/50, unequally weighted parts)

Consider the Gaussian random vector $X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^T$ with zero mean and covariance matrix

$$K_X = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} .$$

(a) (3 points) Find $P[|X_1 + X_2| \geq 3]$. Express your answer in terms of the function

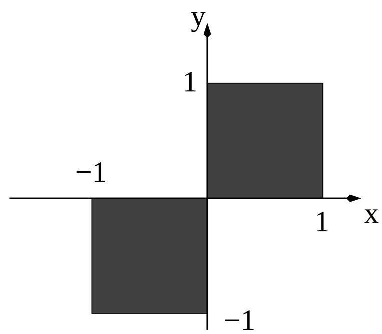
$$Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du .$$

(b) (3 points) Find the minimum mean squared error (MMSE) estimator of random variable X_1 based on observations of the random variable X_2 (i.e., find $\widehat{X}_{1,MMSE}(x_2)$). Also, calculate the associated mean square error.

(c) (4 points) Find the minimum mean squared error (MMSE) estimator of random variable X_1 based on observations of random variables X_2 and X_3 (i.e., find $\widehat{X}_{1,MMSE}(x_2, x_3)$). Also, calculate the associated mean square error.

Problem 3 (12/50, equally weighted parts)

Random variables X and Y have joint pdf $f_{X,Y}(x, y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X?

(b) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X given the observation $Y = y$. What is $E[(X - \widehat{X}_{MMSE}(Y))^2]$, the associated mean squared error?

(c) Determine $\widehat{X}_{LMMSE}(y)$, the linear minimum mean square error estimator for X given the observation $Y = y$. What is $E[(X - \widehat{X}_{LMMSE}(Y))^2]$, the associated mean squared error?

(d) Consider a nonlinear estimator of the form $\widehat{X}(y) = \alpha y^2 + \beta y + \gamma$. Choose the constants α, β and γ such that the mean squared error $E[(X - \widehat{X}(Y))^2]$ is minimized. Find the associated mean squared error.

Problem 4 (8/50, equally weighted parts)

Let Θ be uniformly distributed in the interval $[0, 2\pi]$ and consider the sequence of random variables

$$X_n = \cos(n\Theta) \text{ for } n = 1, 2, ...$$

The following **two** parts can be answered **independently**.

Part A. Does X_n converge almost surely (a.s.) as n tends to infinity? Justify your answer.

Part B. Does X_n converge in the mean square (m.s.) sense as n tends to infinity? Justify your answer. [**Hint:** Recall that $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ and $\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$.]

Problem 5 (equally weighted parts) (8/50)

This problem has **two independent** parts.

Part A. Let $X_1, X_2, ...$ be a sequence of independent, identically distributed (i.i.d.) random variables such that

$$f_{X_i}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 , \\ 0 & \text{otherwise.} \end{cases}$$

Find a good bound for $P\left[\frac{\sum_{i=1}^{n} X_i}{n} \geq 2\lambda\right]$ (you can assume that $\lambda \geq \sqrt{\frac{1}{2}}$).

Part B. Suppose that random variable X satisfies $E[X^4] = 100$. Derive a non-trivial upper bound on $P\left[|X| \geq 10\right]$.

Bonus Problem (5/50)

Suppose that X_0 is a scalar random variable with zero mean and known variance $\mathrm{cov}(X_0) = E[X_0^2] = \sigma_{X_0}^2$ and that $W_0, W_1, W_2, \ldots$ is a sequence of independent, identically distributed (i.i.d.) random variables with zero mean and known variance $\mathrm{cov}(W_i) = E[W_i^2] = \sigma_W^2$. Furthermore, assume that X_0 and the W_i's are independent. Given that

$$X_{k+1} = \alpha X_k + \beta W_k$$

for $k = 0, 1, 2, \ldots$, find a recursive method for computing the variance of X_k. Also, derive conditions under which the mean and variance of X_k converge to steady state values. What are these steady state values?