University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 434: RANDOM PROCESSES

Spring 2004

Problem Set 7

Karhunen-Loeve Expansions; Random Processes through Linear Systems; Wiener Filtering

Issued: Friday, April 30th **Due:** Never (but included in the Final Exam)

Reading from Hajek: Chapter 6 (in particular, Sections 6.1–6.3); Chapter 7 (in particular, Sections 7.1–7.4).

Reading from Stark and Woods: Chapter 8, Section 8.5; Chapter 7, Sections 7.3–7.5; Chapter 9, Section 9.3.

Problem 7.1

From Hajek, Chapter 5: Problem 8.

Problem 7.2

Let N_t, $t \geq 0$, be a Poisson random process with rate $\lambda > 0$.

(a) In which sense is N_t continuous? Almost surely? In the mean-square sense? In probability? In distribution?

(b) Let $T > 0$ and describe the Karhunen-Loeve expansion for the centered process $N_t - \lambda t$ in the interval $[0, T]$ (i.e., provide the basis functions and the corresponding eigenvalues).

Problem 7.3 (optional)

From Hajek, Chapter 5: Problem 9.

Problem 7.4

From Hajek, Chapter 6: Problems 1 and 2.

Problem 7.5

From Hajek, Chapter 6: Problems 3 and 4.

Problem 7.6

Let X_t be an arbitrary stochastic process and $\widehat{X}_t$ be the MMSE optimal linear estimate of the current value of X_t based on just two past values X_{t_1} and X_{t_2} for $t_1 < t_2 < t$. What are the necessary and sufficient conditions on $R_X(t,s)$ such that $\widehat{X}_t$ only depends on the most recent value X_{t_2}?

Problem 7.7

From Hajek, Chapter 7: Problems 1 and 3.

Problem 7.8 (optional)

From Hajek, Chapter 7: Problem 2.

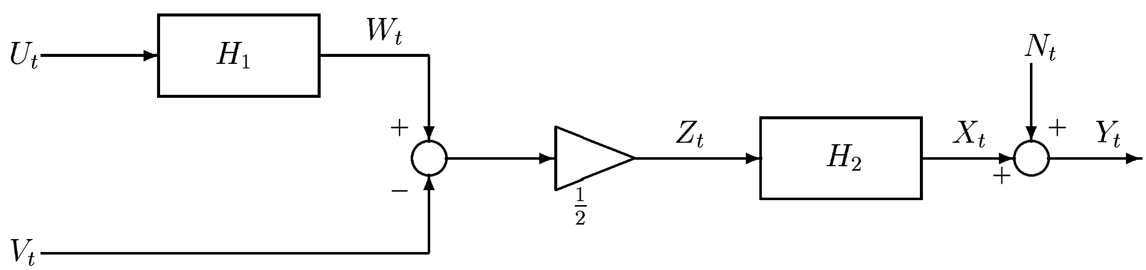
Problem 7.9

You are given three wide-sense stationary (WSS) random processes U_t, V_t and N_t which are zero mean, mutually uncorrelated and have power spectral densities as given below

$$S_U(\omega) = 1 \,, \quad S_V(\omega) = 4 \,, \quad S_N(\omega) = \frac{3}{1+\omega^2} \,.$$

We introduce four other random processes, Y_t, X_t, Z_t and W_t, related to U_t, V_t and N_t via the block diagram below where

$$H_1(\omega) = \frac{2}{1+j\omega} \,, \quad H_2(\omega) = \frac{1}{2+j\omega} \,.$$



(a) Obtain expressions for the individual power spectral densities of Y_t and X_t, and the cross spectral density between Y_t and X_t.

(b) Determine the filter transfer function $H_3(\omega)$ such that its output $\widehat{X}_t$ provides a linear MMSE estimate for X_t based on the output process Y_t.

Problem 7.10

Let X_t be a real-valued, zero-mean stationary Gaussian process with $R_X(\tau) = e^{-|\tau|}$. Let $\alpha > 0$ and suppose that X_0 is estimated by $\widehat{X}_0 = c_1 X_{-\alpha} + c_2 X_\alpha$, where the constants c_1 and c_2 are chosen to minimize the mean square error (MSE).

(a) Find the optimal choice for c_1 and c_2 and the corresponding MSE $E[(X_0 - \widehat{X}_0)^2]$.

(b) Use the orthogonality principle to show that $\widehat{X}_0$ as defined above is the MMSE estimator of X_0 given X_s for $|s| \geq \alpha$.

Problem 7.11

From Hajek, Chapter 7: Problems 4 and 5.

Problem 7.12 (optional)

From Hajek, Chapter 7: Problems 6 and 7.