ECE 534 FA26 HW5 Solutions to Odd-Numbered Problems

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The solutions to even-numbered problems can be found in [1, Chapter 12].

Problem 4.5 A random line.

Solution.

(a) Three sample paths are shown in Figure 1. The graph of each sample path is a line with positive intercept at $t = 0$ and negative slope.

(b) We claim that X is not a Markov process.

By the definition of a Markov process (in [1, Definition 4.6]), if X were Markov, then we must have $(X_{-1}, X_0) - X_0 - X_1$. That is, for every Borel set $B \subseteq \mathbb{R}$, we have

$$P(X_1 \in B \mid X_{-1}, X_0) = P(X_1 \in B \mid X_0) \quad \text{a.s.}$$

See the paragraph above [1, Definition 4.6] for a definition of these notations.

Take $B = (-\infty, 0)$. Since

$$X_{-1} = R + S, \qquad X_0 = R, \qquad X_1 = R - S,$$

we have

$$\begin{aligned} P(X_1 \in B \mid X_{-1}, X_0) &= E[\mathbb{1}_{\{R-S<0\}} \mid R+S, R] \\ &= \mathbb{1}_{\{R-S<0\}} \\ &= \mathbb{1}_{\{S>R\}}, \end{aligned}$$

where the second equality follows from the fact that $\mathbb{1}_{\{R-S<0\}} = \mathbb{1}_{\{2R-(R+S)<0\}}$ is a measurable function of $(R+S, R)$ and Example 3.3.

On the other hand, we claim that

$$P(X_1 \in B \mid X_0) = 1 - F_S(R),$$

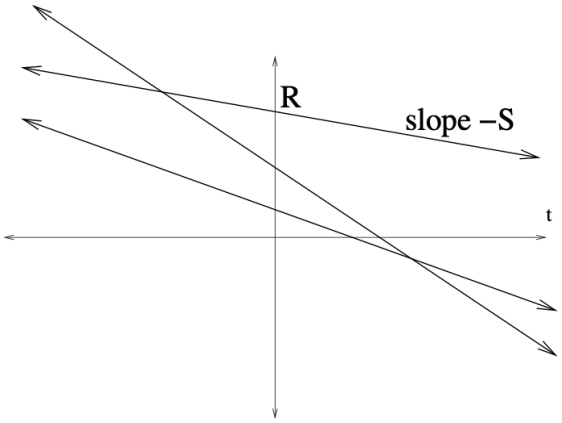


Fig. 1: Three sample paths.

where F_S is the CDF of S. To see this, let $g : \mathbb{R} \to \mathbb{R}$ be any Borel measurable function such that $E[g(R)^2] < \infty$. By the independence of R and S, we have

$$\begin{aligned} &E[g(R)\mathbb{1}_{\{R-S<0\}}] \\ &= \int_0^\infty \int_0^\infty g(r)\mathbb{1}_{\{s>r\}} f_R(r) f_S(s)\, ds\, dr \\ &= \int_0^\infty g(r) f_R(r) \left(\int_r^\infty f_S(s)\, ds \right) dr \\ &= \int_0^\infty g(r)(1 - F_S(r)) f_R(r)\, dr \\ &= E[g(R)(1 - F_S(R))]. \end{aligned}$$

Note that the integrals are well-defined since by Cauchy-Schwarz we have $E[|g(R)|] \le \sqrt{E[g(R)^2]} < \infty$. Then, note that $1 - F_S(R)$ is a bounded measurable function of R, and thus has a finite second moment. By the characterization of conditional expectation, we know that

$$E[\mathbb{1}_{\{R-S<0\}} \mid R] = 1 - F_S(R) \quad \text{a.s.}$$

In fact, we can express $F_S(R)$ explicitly. Since S is Rayleigh with parameter σ_S^2, we have

$$1 - F_S(R) = \exp\left(-\frac{R^2}{2\sigma_S^2} \right).$$

As $R > 0$ almost surely, we have

$$0 < \exp\left(-\frac{R^2}{2\sigma^2} \right) < 1 \quad \text{a.s.}$$

However, $\mathbb{1}_{\{S>R\}}$ takes values only in $\{0, 1\}$. Therefore, we have

$$\mathbb{1}_{S>R} \neq \exp\left(-\frac{R^2}{2\sigma^2} \right) \quad \text{a.s.},$$

or equivalently,

$$P(X_1 \in B \mid X_{-1}, X_0) \neq P(X_1 \in B \mid X_0) \quad \text{a.s.}$$

These arguments show that X is not a Markov process.

Remark. Intuitively, since X_1 is completely determined by X_0 and X_{-1} by $X_1 = 2X_0 - X_{-1}$, while there is still some uncertainty of X_1 given X_0, so we can expect that the condition $(X_{-1}, X_0) - X_0 - X_1$ does not hold. However, note that we cannot directly use the density-version criterion in [1, Equation (4.9)] since $((X_{-1}, X_0), X_0, X_1)$ does not have a joint density (again, due to the fact that X_1 is completely determined by X_0 and X_{-1}).

(c) No, X does not have independent increment. For example, the increments $X_1 - X_0$ and $X_2 - X_1$ are both $-S$, but $-S$ is not independent of itself. The reason is that if a random variable Y is independent of itself, then Y is almost surely a constant (to see this, for each $c \in \mathbb{R}$ we have $P(Y \le c) = P(Y \le c, Y \le c) = P(Y \le c)^2$ and thus $F_Y(c) \in \{0, 1\}$), which is not the case here since S follows a Rayleigh distribution. Therefore, $X_1 - X_0$ is not independent of $X_2 - X_1$, and thus X does not have independent increment.
(d) The triangle has height R and base of width R/S, so its area is

$$A = \frac{R^2}{2S}.$$

Since R and S are independent,

$$E[A] = \frac{1}{2} E[R^2] E\left[\frac{1}{S} \right].$$

Since R^2 has the same distribution as the sum of the squares of two independent $N(0, \sigma_R^2)$ random variables, we have

$$E[R^2] = 2\sigma_R^2.$$

On the other hand, we have

$$\begin{aligned} E\left[\frac{1}{S}\right] &= \int_0^\infty \frac{1}{s}\frac{s}{\sigma_S^2}\exp\left(-\frac{s^2}{2\sigma_S^2}\right) ds \\ &= \frac{\sqrt{2\pi}}{\sigma_S}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma_S^2}}\exp\left(-\frac{s^2}{2\sigma_S^2}\right) ds \\ &= \frac{1}{\sigma_S}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Combining the above yields

$$E[A] = \frac{\sigma_R^2}{\sigma_S}\sqrt{\frac{\pi}{2}}.$$

Problem 4.7: Some probabilities for a Brownian motion

(a) The event in question can be written as

$$\begin{aligned} \left\{W_3 \geq \frac{W_2+W_4}{2}+1\right\} &= \{2W_3 - W_2 - W_4 \geq 2\} \\ &= \{(W_3 - W_2) - (W_4 - W_3) \geq 2\}. \end{aligned}$$

Since $W_3 - W_2$ and $W_4 - W_3$ are independent $N(0,1)$ random variables, their difference has the $\mathcal{N}(0,2)$ distribution. Therefore, we have

$$\begin{aligned} P\left\{W_3 \geq \frac{W_2+W_4}{2}+1\right\} &= P\left\{\mathcal{N}(0,2) \geq 2\right\} \\ &= P\left\{\mathcal{N}(0,1) \geq \frac{2}{\sqrt{2}}\right\} \\ &= Q(\sqrt{2}). \end{aligned}$$

Alternative method. Let $Z = -W_2 + 2W_3 - W_4$. Since W is a Gaussian process, Z is Gaussian. Clearly $E[Z] = 0$. Moreover, using $\mathrm{Cov}(W_s, W_t) = \min\{s,t\}$, we get

$$\begin{aligned} \mathrm{Var}(Z) &= \mathrm{Var}(W_2) + 4\,\mathrm{Var}(W_3) + \mathrm{Var}(W_4) \\ &\quad - 4\,\mathrm{Cov}(W_2, W_3) - 4\,\mathrm{Cov}(W_3, W_4) + 2\,\mathrm{Cov}(W_2, W_4) \\ &= 2 + 12 + 4 - 8 - 12 + 4 \\ &= 2. \end{aligned}$$

Thus $Z \sim \mathcal{N}(0,2)$, and the required probability is $P(Z \geq 2) = Q(\sqrt{2})$.

(b) For every $t \geq 1$, $W_t/\sqrt{t} \sim N(0,1)$. Hence W_t^2/t has the same distribution as Z^2, where $Z \sim \mathcal{N}(0,1)$; in particular, this distribution does not depend on t. It follows that the limiting distribution of $\frac{W_t^2}{t}$ is also the same as Z^2.

The PDF of Z^2 is found as follows. For $c > 0$, we have $P\left\{Z^2 > c\right\} = 2P\left\{Z > \sqrt{c}\right\} = 2Q(\sqrt{c})$. Taking the derivative with respect to c and using the chain rule and definition of Q yields for $c > 0$ that

$$f_{Z^2}(c) = -\frac{dP\left\{Z^2 > c\right\}}{dc} = 2\frac{1}{2\sqrt{c}}\frac{1}{\sqrt{2\pi}}e^{-c/2} = \frac{1}{\sqrt{2\pi c}}e^{-c/2}.$$

At the same time, it is clear that for $c < 0$ we have $P\left\{Z^2 > c\right\} = 1$, and thus $f_{Z^2}(c) = 0$ for $c < 0$. Since the value of a PDF at a single point does not affect the distribution, we may as well set $f_{Z^2}(0) = 0$. In summary, the PDF of Z^2, which is the PDF of the limiting distribution of interest, is given by

$$f_{Z^2}(c) = \begin{cases} \frac{1}{\sqrt{2\pi c}}e^{-c/2}, & \text{if } c > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. Z^2 has the chi-square distribution with one degree of freedom, or, equivalently, the gamma distribution with parameter $1/2$.

Problem 4.11: Some Poisson process calculations

(a) Since N_1 and $N_2 - N_1$ are independent $\mathrm{Poi}(\lambda)$ random variables, we have

$$\begin{aligned} P(N_1 \geq 1, N_2 = 2) &= P(N_1 = 1, N_2 - N_1 = 1) + P(N_1 = 2, N_2 - N_1 = 0) \\ &= (\lambda e^{-\lambda})^2 + \left(\frac{\lambda^2 e^{-\lambda}}{2}\right) e^{-\lambda} \\ &= \frac{3\lambda^2 e^{-2\lambda}}{2}. \end{aligned}$$

Also, since $N_2 \sim \mathrm{Poi}(2\lambda)$, we have

$$P(N_2 = 2) = \frac{(2\lambda)^2 e^{-2\lambda}}{2}.$$

Combining these two results, we get

$$P(N_1 \geq 1 \mid N_2 = 2) = \frac{3\lambda^2 e^{-2\lambda}/2}{(2\lambda)^2 e^{-2\lambda}/2} = \frac{3}{4}.$$

(b) Since $P(N_1 \geq 1) = 1 - e^{-\lambda}$, together with the fact that $P(N_1 \geq 1, N_2 = 2) = 3\lambda^2 e^{-2\lambda}/2$ (which we have calculated in Part (a)), we have

$$P(N_2 = 2 \mid N_1 \geq 1) = \frac{P(N_1 \geq 1, N_2 = 2)}{P(N_1 \geq 1)} = \frac{3\lambda^2 e^{-2\lambda}}{2(1 - e^{-\lambda})}.$$

(c) Yes, X is a time-homogeneous Markov process. Write $\mathbb{Z}_{\geq 0}$ for the set of nonnegative integers. Observe that for each positive integer n, each $0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1}$, and each $i_1, \ldots, i_{n+1} \in \mathbb{Z}_{\geq 0}$ satisfying $P(X_{t_n} = i_n^2, \ldots, X_{t_1} = i_1^2) > 0$ (equivalently $P(N_{t_n} = i_n, \ldots, N_{t_1} = i_1) > 0$), we have

$$\begin{aligned} &P\left(X_{t_{n+1}} = i_{n+1}^2 \middle| X_{t_n} = i_n^2, \ldots, X_{t_1} = i_1^2\right) \\ &= P\left(N_{t_{n+1}} = i_{n+1} \middle| N_{t_n} = i_n, \ldots, N_{t_1} = i_1\right) \\ &= P\left(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n \middle| N_{t_n} = i_n, \ldots, N_{t_1} = i_1\right) && (1) \\ &= P\left(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n \middle| N_{t_n} - N_{t_{n-1}} = i_n - i_{n-1}, \ldots, N_{t_2} - N_{t_1} = i_2 - i_1, N_{t_1} - N_0 = i_1\right) && (2) \\ &= P\left(N_{t_{n+1}} - N_{t_n} = i_{n+1} - i_n\right) && (3) \\ &= P\left(\mathrm{Poi}(\lambda(t_{n+1} - t_n)) = i_{n+1} - i_n\right) \\ &= \begin{cases} \dfrac{(\lambda(t_{n+1} - t_n))^{i_{n+1} - i_n} e^{-\lambda(t_{n+1} - t_n)}}{(i_{n+1} - i_n)!}, & \text{if } i_{n+1} \geq i_n, \\ 0, & \text{if } i_{n+1} < i_n, \end{cases} && (4) \end{aligned}$$

where (1) follows from the fact that the condition $N_{t_n} = i_n$ has been given, (2) follows from the fact that $N_0 = 0$ a.s., and (3) follows from the independent increment property of N. In fact, a similar calculation shows that $P\left(X_{t_{n+1}} = i_{n+1}^2 \middle| X_{t_n} = i_n^2\right)$ is also equal to (4). Therefore, we have $(X_{t_1}, \ldots, X_{t_n}) - X_{t_n} - X_{t_{n+1}}$, and thus X is a Markov process with state space

$$\mathcal{S} \coloneqq \{i^2 \ : \ i \in \mathbb{Z}_{\geq 0}\} = \{0, 1, 4, 9, \ldots\}.$$

Furthermore, since the transition probability in (4) only depends on $t_{n+1} - t_n$, we can further conclude that X is time-homogeneous. To be more precise, the transition probabilities of X for nonnegative integers i, j and $\tau \geq 0$ are given by

$$\begin{aligned} p_{i^2, j^2}(\tau) &= P(X_{t+\tau} = j^2 \mid X_t = i^2) \\ &= P(N_{t+\tau} = j \mid N_t = i) \\ &= \begin{cases} \dfrac{(\lambda\tau)^{j-i} e^{-\lambda\tau}}{(j-i)!}, & j \geq i, \\ 0, & j < i, \end{cases} \end{aligned}$$

where by convention we denote $0^0 = 1$ in the case $\tau = 0$ (i.e. $p_{i^2, j^2}(0)$ equals 1 if $i = j$ and is 0 if $i \neq j$).

Problem 4.17: Invariance of properties under transformations

Recall that $Y_n = X_n^2$ and $Z_n = X_n^3$ for all $n \in \mathbb{Z}$.

(a) FALSE. Let X be the time-homogeneous Markov process on $\{-1, 0, 1\}$ that cycles deterministically from -1 to 0 to 1 and back to -1, with X_0 uniformly distributed on $\{-1, 0, 1\}$. Its one-step transition probability matrix, in the state order $(-1, 0, 1)$, is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The sample paths of Y are the three phase shifts of the periodic sequence $\ldots, 1, 0, 1, 1, 0, 1, \ldots$. Therefore, for any n, we have

$$P(Y_{n+1} = 1 \mid Y_n = 1) = \frac{1}{2}, \qquad P(Y_{n+1} = 1 \mid Y_n = 1, Y_{n-1} = 0) = 1.$$

Hence Y is not Markov.

(b) TRUE. For the case where X takes values in a discrete space, the calculation is similar to Problem 4.11(c).

We now prove the general case using the characterization of conditional probability, as outlined in [1, Page 126]. To this end, we will need the following lemma:

Lemma 1. *Let U be a random variable taking values in $\mathbb{R}^k$. Assume $f : \mathbb{R}^k \to \mathbb{R}^k$ is a bijection between $\mathbb{R}^k$ and itself such that both f and its inverse f^{-1} are Borel measurable, and define $V \coloneqq f(U)$. Then, for any event F we have*

$$P(F|U) = P(F|V) \text{ a.s.}$$

Proof of Lemma 1. Let $h : \mathbb{R}^k \to \mathbb{R}$ be any Borel measurable function such that $h(U)$ has finite second moment. Then, define $g : \mathbb{R}^k \to \mathbb{R}$ by $g(x) \coloneqq h(f^{-1}(x))$, which is also a measurable function. Note that we now have $h(U) = h(f^{-1}(V)) = g(V)$. It follows that

$$\begin{aligned} E[h(U)\mathbb{1}_F] &= E[g(V)\mathbb{1}_F] \\ &= E[g(V)P(F|V)], \end{aligned} \tag{5}$$

where the last equality follows from the fact that $g(V) = h(U)$ has finite second moment by construction and Property (2) of conditional probability on [1, Page 126]. Furthermore, recall that we can write $P(F|V) = \eta(V)$ for some measurable function η by Property (1) of conditional probability on [1, Page 126]. Then, from (5) we have

$$\begin{aligned} E[h(U)\mathbb{1}_F] &= E[g(V)\eta(V)] \\ &= E[h(U)\eta(f(U))]. \end{aligned} \tag{6}$$

Since $\eta(f(U)) = \eta(V) = P(F|V)$ has finite second moment (in fact bounded) and $\eta(f(U))$ is a measurable function of U, from (6) and the characterization property of conditional expectation (i.e. Properties (1) and (2)) we can deduce that

$$P(F|U) = \eta(f(U)) \text{ a.s.},$$

or equivalently, $P(F|U) = P(F|V)$ a.s. □

Now let n be any positive integer and $t_1, \ldots, t_{n+1} \in \mathbb{Z}$ such that $t_1 < \cdots < t_{n+1}$. Also let B be any Borel subset of $\mathbb{R}$. Define $B' \coloneqq \ell^{-1}(B)$, where ℓ denotes the mapping $t \mapsto t^3$. Note that B' is also measurable since ℓ is a measurable function, and that for all $x \in \mathbb{R}$ we have $x^3 \in B$ iff $x \in B'$. Now we have almost surely

$$\begin{aligned} P\left(Z_{t_{n+1}} \in B \middle| Z_{t_1}, \ldots, Z_{t_n}\right) &= P\left(X_{t_{n+1}}^3 \in B \middle| X_{t_1}^3, \ldots, X_{t_n}^3\right) \\ &= P\left(X_{t_{n+1}} \in B' \middle| X_{t_1}, \ldots, X_{t_n}\right) && (7) \\ &= P\left(X_{t_{n+1}} \in B' \middle| X_{t_n}\right) && (8) \\ &= P\left(X_{t_{n+1}}^3 \in B \middle| X_{t_n}^3\right) && (9) \\ &= P\left(Z_{t_{n+1}} \in B \middle| Z_{t_n}\right), && (10) \end{aligned}$$

where

- Equation (7) is obtained from the definition of B' and an application of Lemma 1 with $k = n$, $U = (X_{t_1}^3, \ldots, X_{t_n}^3)$, and $f(y_1, \ldots, y_n) = (y_1^{1/3}, \ldots, y_n^{1/3})$ (correspondingly $V = (X_{t_1}, \ldots, X_{t_n})$).
- Equation (8) follows from the assumption that X is Markov and thus $(X_{t_1}, \ldots, X_{t_n}) - X_{t_n} - X_{t_{n+1}}$.
- Equation (9) is obtained from the definition of B' again and an application of Lemma 1 with $k = 1$, $U = X_{t_n}$, and $f(y) = y^3$ (correspondingly $V = X_{t_n}^3$).

Since (10) holds for all Borel sets B, we can deduce that $(Z_{t_1}, \ldots, Z_{t_n}) - Z_{t_n} - Z_{t_{n+1}}$. Lastly, since $n \geq 1$ and $t_1 < \cdots < t_{n+1}$ are arbitrary, we can conclude that Z is Markov.

(c) FALSE. Let X have the three equally likely phase shifts of the periodic sequence

$$\ldots, -1, 1, 1, -1, 1, 1, \ldots.$$

Then

$$P(X_{n+1} = -1 \mid X_n = 1) = \frac{1}{2}, \qquad P(X_{n+1} = -1 \mid X_n = 1, X_{n-1} = 1) = 1.$$

Thus X is not Markov. However, $Y_n = X_n^2 = 1$ for all n, so Y is a Markov process.

(d) TRUE. If X is stationary, then for any positive integer n, for any $t_1, \ldots, t_n \in \mathbb{Z}$, and for any $s \in \mathbb{Z}$, the two random vectors $(X_{s+t_1}, \ldots, X_{s+t_n})$ and $(X_{t_1}, \ldots, X_{t_n})$ have the same distribution, denoted as

$$(X_{s+t_1}, \ldots, X_{s+t_n}) \overset{(d)}{=} (X_{t_1}, \ldots, X_{t_n}).$$

It follows that we also have

$$(Y_{s+t_1}, \ldots, Y_{s+t_n}) \overset{(d)}{=} (Y_{t_1}, \ldots, Y_{t_n}).$$

To see this, for any $c_1, \ldots, c_n \geq 0$, we have

$$\begin{aligned} P\left(X_{s+t_1}^2 \leq c_1, \ldots, X_{s+t_n}^2 \leq c_n\right) &= P\left(-\sqrt{c_1} \leq X_{s+t_1} \leq \sqrt{c_1}, \ldots, -\sqrt{c_n} \leq X_{s+t_n} \leq \sqrt{c_n}\right) \\ &= P\left(-\sqrt{c_1} \leq X_{t_1} \leq \sqrt{c_1}, \ldots, -\sqrt{c_n} \leq X_{t_n} \leq \sqrt{c_n}\right) \\ &= P\left(X_{t_1}^2 \leq c_1, \ldots, X_{t_n}^2 \leq c_n\right), \end{aligned}$$

where the second equality follows from the assumption that $(X_{s+t_1}, \ldots, X_{s+t_n}) \overset{(d)}{=} (X_{t_1}, \ldots, X_{t_n})$. At the same time, if any one of $c_1, \ldots, c_n$ is strictly negative, then both $P\left(X_{s+t_1}^2 \leq c_1, \ldots, X_{s+t_n}^2 \leq c_n\right)$ and $P\left(X_{t_1}^2 \leq c_1, \ldots, X_{t_n}^2 \leq c_n\right)$ are equal to 0. These arguments show that $(Y_{s+t_1}, \ldots, Y_{s+t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ have the same joint CDF and thus have the same distribution. Since the arguments above hold for all $n \geq 1$, $t_1, \ldots, t_n \in \mathbb{Z}$, and $s \in \mathbb{Z}$, we can conclude that Y is stationary.

(e) FALSE. Consider the deterministic process defined by $X_0 = -1$ and $X_n = 1$ for all $n \neq 0$. Then X is not stationary, whereas $Y_n = X_n^2 = 1$ for every n, so Y is stationary.

(f) FALSE. Let $(X_n : n \in \mathbb{Z})$ be independent, mean-zero, variance-one random variables with finite but unequal fourth moments. Then X is wide-sense stationary, since

$$E[X_n] = 0, \qquad E[X_n X_m] = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

But $E[Y_n^2] = E[X_n^4]$ depends on n, so Y is not wide-sense stationary. For example, take $X_0 \sim N(0, 1)$ and let the other X_n be independent random variables taking -1 and 1 with equal probabilities; then $E[Y_0^2] = E[X_0^4] = 3$ while $E[Y_n^2] = E[X_n^4] = 1$ for $n \neq 0$.

(g) FALSE. Let $X_n = 0$ for $n \leq 0$ and $X_n = W_1 + \cdots + W_n$ for $n \geq 1$, where the W_n are independent and $P(W_n = 0) = P(W_n = 1) = 1/2$. Then X has independent increments. However,

$$P(Y_2 - Y_1 = 3 \mid Y_1 - Y_0 = 1) = \frac{1}{2}, \qquad P(Y_2 - Y_1 = 3 \mid Y_1 - Y_0 = 0) = 0.$$

Thus the increments $Y_2 - Y_1$ and $Y_1 - Y_0$ are dependent, and Y does not have independent increments.

(h) FALSE. Let $X_n = 0$ for $n \leq 0$ and $X_n = B$ for $n \geq 1$, where $P(B = -1) = 2/3$ and $P(B = 2) = 1/3$. Since

$$E[B] = -\frac{2}{3} + \frac{2}{3} = 0,$$

it can be checked that X is a martingale. To be more precise, we first check that $E\left[|X_n|\right] = E\left[0\right] = 0 < \infty$ for $n \leq 0$ and $E\left[|X_n|\right] = E\left[|B|\right] = \frac{4}{3} < \infty$ for $n \geq 1$. Then, for any positive integer n and any integers $t_1 < \cdots < t_{n+1}$, we have the following:

- If $t_{n+1} \leq 0$, then $X_{t_1} = \cdots = X_{t_{n+1}} = 0$ and $E\left[X_{t_{n+1}} | X_{t_1}, \ldots, X_{t_n}\right] = 0 = X_{t_n}$.
- If $t_{n+1} \geq 1$ but $t_n \leq 0$, then $X_{t_1} = \cdots = X_{t_n} = 0$, and $E\left[X_{t_{n+1}} | X_{t_1}, \ldots, X_{t_n}\right] = E\left[B\right] = 0 = X_{t_n}$.
- If $t_n \geq 1$, then $E\left[X_{t_{n+1}} | X_{t_1}, \ldots, X_{t_n}\right] = B = X_{t_n}$.

However, $Z_n = X_n^3$ is not a martingale, because

$$E[Z_1 \mid Z_0] = E[B^3] = -\frac{2}{3} + \frac{8}{3} = 2 \neq 0 = Z_0.$$

REFERENCES

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf