ECE 534 FA26 HW4 Solutions to Odd-Numbered Problems

Updated on October 31, 2026

We include the solution to Problem 2.32(b) in this solution set. The solution to this subproblem in [1, Chapter 12] has a few typos. We have fixed those typos and added a few more explanations here.

The solutions to Problem 2.32(a) and the remaining even-numbered problems can be found in the aforementioned chapter.

Problem 2.31 A large deviation.

Solution. The probability of interest can be rewritten as

$$P\left(\frac{X_1^2+\cdots+X_n^2}{n} \geq 2\right).$$

Since $E\left[X_1^2\right] = \mathrm{Var}\left(X_1\right) = 1 < 2$, Cramér's theorem [1, Theorem 2.19] implies that Chernoff bound gives the exact exponent. That is, we have $b = \ell(2)$ (the definition of ℓ is in [1, Equation (2.12)]), which we now compute. For $a > 0$, we have

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(\frac{1}{2a})}} dx = \sqrt{\frac{\pi}{a}},$$

so the moment generating function of X_1^2 is given by

$$M(\theta) = E\left[e^{\theta X_1^2}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2(\frac{1}{2}-\theta)} dx = (1-2\theta)^{-\frac{1}{2}},$$

which is well-defined for $\theta < \frac{1}{2}$. It follows that the log moment generating function of X_1^2 is $\ln M(\theta) = -\frac{1}{2}\ln(1-2\theta)$, and thus

$$\ell(a) = \max_{\theta<1/2}\left\{\theta a + \frac{1}{2}\ln(1-2\theta)\right\}.$$

By differentiation, we found that

$$\begin{aligned}\theta^* &= \operatorname*{argmax}_{\theta<1/2}\left\{\theta a + \frac{1}{2}\ln(1-2\theta)\right\} \\ &= \frac{1}{2}\left(1-\frac{1}{a}\right),\end{aligned}$$

which gives

$$\ell(a) = \frac{1}{2}\left(a - \ln a - 1\right).$$

The exponent of interest b is then

$$b = \ell(2) = \frac{1}{2}(1-\ln 2) \approx 0.1534264097.$$

The numerical value of interest is

$$e^{-100b} \approx 2.17 \times 10^{-7}.$$

Problem 2.32 Some large deviations.

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Solution.

(b) The probability in question is equal to

$$P\{X_1 + \cdots + X_n > 0\},$$

where $X_k = U_k - cU_{n+k}$ for $1 \le k \le n$. Then $X_1, \ldots, X_n$ are i.i.d. and

$$E[X_k] = \frac{1-c}{2}.$$

Therefore, if $c > 1$, we have $0 > \frac{1-c}{2} = E[X_k]$, and thus the required bound is provided by Chernoff's inequality applied to $X_1, \ldots, X_n$.

If $c = 1$, the probability is exactly 0.5 for all n and does not satisfy the required bound for any $b > 0$. To see this, let $S_n \coloneqq U_1 + \cdots + U_n$ and $T_n \coloneqq U_{n+1} + \cdots + U_{2n}$. Then S_n and T_n are i.i.d. continuous random variables, so $P\{S_n = T_n\} = 0$, and by symmetry $P\{S_n > T_n\} = P\{S_n < T_n\}$. Therefore, we have $P\{S_n > T_n\} = 0.5$.

If $0 \le c < 1$, then since $T_n \ge 0$ a.s., we have

$$P\{S_n > cT_n\} \ge P\{S_n > T_n\} = \frac{1}{2},$$

so the required bound cannot hold either.

In summary, such $b > 0$ exists if and only if $c > 1$.

Problem 2.33 Sums of independent Cauchy random variables.

Solution.

(a) Find the characteristic function of S_n/n^θ for a constant θ.

The characteristic function of S_n/n^θ, denoted as $\Phi_{S_n/n^\theta}(u)$, satisfies

$$\begin{aligned}
\Phi_{S_n/n^\theta}(u) &= E\left[\exp\left(ju\frac{S_n}{n^\theta}\right)\right] \\
&= E\left[\exp\left(ju\frac{X_1}{n^\theta}\right)\cdots\exp\left(ju\frac{X_n}{n^\theta}\right)\right] \\
&= E\left[\exp\left(ju\frac{X_1}{n^\theta}\right)\right]\cdots E\left[\exp\left(ju\frac{X_n}{n^\theta}\right)\right] \\
&= E\left[\exp\left(ju\frac{X_1}{n^\theta}\right)\right]^n,
\end{aligned}$$

where the third equality follows from the independence of $X_1, \ldots, X_n$, and the fourth equality follows from the fact that $X_1, \ldots, X_n$ have the same distribution. Note that

$$\begin{aligned}
E\left[\exp\left(ju\frac{X_1}{n^\theta}\right)\right] &= \Phi\left(\frac{u}{n^\theta}\right) \\
&= \exp\left(-\frac{|u|}{n^\theta}\right),
\end{aligned}$$

where $\Phi(u) = \exp(-|u|)$ is the characteristic function of the standard Cauchy distribution as given in the problem description. Therefore, we have

$$\boxed{\Phi_{S_n/n^\theta}(u) = \exp\left(-\frac{|u|}{n^\theta}\right)^n = \exp(-|u|n^{1-\theta}).}$$

(b) Does S_n/n converge in distribution?

Setting $\theta = 1$ in the result from Part (a), we get

$$\Phi_{S_n/n}(u) = e^{-n^{1-1}|u|} = e^{-|u|} = \Phi(u) \quad \text{for all } n.$$

This is the characteristic function of a standard Cauchy for *every* n. In particular, for all $u \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \Phi_{S_n/n}(u) = \Phi(u).$$

Since pointwise convergence of characteristic functions implies convergence in distribution (see [1, Proposition 2.6]), we have

$$\boxed{\frac{S_n}{n} \xrightarrow{d} \text{Cauchy}(0,1).}$$

(c) Does S_n/n^2 converge in distribution?

Setting $\theta = 2$ in the result from Part (a), we get for each $u \in \mathbb{R}$ that

$$\Phi_{S_n/n^2}(u) = e^{-n^{-1}|u|} \xrightarrow[n\to\infty]{} e^0 = 1.$$

The limit 1 is the characteristic function of the constant random variable 0. Similar to Part (b), we can deduce that

$$\boxed{\frac{S_n}{n^2} \xrightarrow{d} 0.}$$

(d) Does $S_n/\sqrt{n}$ converge in distribution?

Setting $\theta = 1/2$,

$$\Phi_{S_n/\sqrt{n}}(u) = e^{-\sqrt{n}\,|u|}.$$

For $u = 0$ this equals 1 for all n. On the other hand, for any $u \neq 0$, we have

$$e^{-\sqrt{n}\,|u|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Hence the pointwise limit is

$$\lim_{n\to\infty} \Phi_{S_n/\sqrt{n}}(u) = \begin{cases} 1, & u = 0, \\ 0, & u \neq 0, \end{cases}$$

which is *not continuous at* $u = 0$. Therefore, the pointwise limit of $\Phi_{S_n/\sqrt{n}}(u)$ is not a valid characteristic function. (You can take for granted the property that the characteristic function of any random variable is necessarily continuous.) We can then conclude that

$$\boxed{\frac{S_n}{\sqrt{n}} \text{ does } \textit{not} \text{ converge in distribution.}}$$

Another approach. Here is another approach that does not use the characteristic function. We use CDF instead. From Part (b) we know that S_n/n and X_1 have the same distribution. We then have for each $c \in \mathbb{R}$ that

$$\begin{aligned} P\left\{\frac{S_n}{\sqrt{n}} \leq c\right\} &= P\left\{\frac{S_n}{n} \leq \frac{c}{\sqrt{n}}\right\} \\ &= P\left\{X_1 \leq \frac{c}{\sqrt{n}}\right\} \\ &= F_{X_1}\left(\frac{c}{\sqrt{n}}\right), \end{aligned}$$

where F_{X_1} is the CDF of X_1. Since the standard Cauchy distribution is a continuous distribution (we define it by describing its density), its CDF is a continuous function. Therefore, we can pass the limit directly inside F_{X_1}, which gives

$$\begin{aligned} \lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \leq c\right\} &= \lim_{n\to\infty} F_{X_1}\left(\frac{c}{\sqrt{n}}\right) \\ &= F_{X_1}\left(\lim_{n\to\infty} \frac{c}{\sqrt{n}}\right) \\ &= F_{X_1}(0) \\ &= \frac{1}{2}. \end{aligned} \tag{1}$$

Now assume for contradiction that $S_n/\sqrt{n}$ converges in distribution to some random variable Y. By (1) and the definition of convergence in distribution, the CDF of Y satisfies $F_Y(c) = \frac{1}{2}$ for all continuity points c of F_Y. Since there can be at most countably many discontinuity points of any CDF, we can choose arbitrarily large continuity points c of F_Y, and on each such point we have $F_Y(c) \leq 1/2$, which contradicts that $\lim_{c\to\infty} F_Y(c) = 1$. This contradiction stems from the assumption that $S_n/\sqrt{n}$ converges in distribution, and therefore $S_n/\sqrt{n}$ does not converge in distribution.

Problem 2.35 Chernoff bound for Gaussian and Poisson random variables

Solution. Assume $c > 0$ instead of $c \geq 0$ in all the subproblems.

(a) See [1, Page 63] for more details on the optimized Chernoff bounds. Here we consider the special case $n = 1$.

Recall that $M_X(\theta) = E\left[e^{\theta X}\right]$ is the moment generating function of X, where $\theta \in \mathbb{R}$. We first calculate $M_X(\theta)$ by

$$\begin{aligned} M_X(\theta) &= E\left[e^{\theta X}\right] \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{\theta x} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+\sigma^2\theta))^2}{2\sigma^2} + \mu\theta + \frac{\sigma^2\theta^2}{2}} dx \\ &= e^{\mu\theta + \frac{\sigma^2\theta^2}{2}}. \end{aligned}$$

Consequently, the log moment generating function of X is $\ln M_X(\theta) = \mu\theta + \frac{\sigma^2\theta^2}{2}$.

Then, following the definition of ℓ in [1, Equation (2.12)], we have

$$\ell(a) = \max_{\theta} \left\{ a\theta - \left(\mu\theta + \frac{\sigma^2\theta^2}{2}\right) \right\} = \frac{(a-\mu)^2}{2\sigma^2},$$

which can be determined by either differentiation or completing the square. Taking $a = \mu + c$ in the optimized Chernoff inequality yields for $c > 0$ that

$$P\{X \geq E[X] + c\} \leq \exp\left(-\frac{c^2}{2\sigma^2}\right).$$

(b) Similarly, we first calculate the moment generating function of Y. A direct calculation shows that

$$\begin{aligned} M_Y(\theta) &= E\left[e^{\theta Y}\right] \\ &= \sum_{k=0}^{\infty} \frac{e^{\theta k}\lambda^k e^{-\lambda}}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{\theta}\lambda)^k}{k!} \\ &= e^{\lambda(e^{\theta}-1)}. \end{aligned}$$

It follows that the log moment generating function of Y is

$$\ln M_Y(\theta) = \ln e^{\lambda(e^{\theta}-1)} = \lambda(e^{\theta} - 1).$$

Therefore, we have

$$\ell(a) = \max_{\theta} \left\{ a\theta - \lambda(e^{\theta} - 1) \right\} = a \ln\left(\frac{a}{\lambda}\right) + \lambda - a,$$

which can be determined by differentiation. Setting $a = \lambda + c$ in the optimized Chernoff inequality yields

$$P\{Y \geq E[Y] + c\} \leq \exp\left(-(\lambda + c)\ln\frac{\lambda + c}{\lambda} + c\right).$$

(c) Using the definition of ψ yields

$$\frac{c^2}{2\lambda}\psi\left(\frac{c}{\lambda}\right) = \lambda g\left(1 + \frac{c}{\lambda}\right) = (\lambda + c)\ln\frac{\lambda + c}{\lambda} - c$$

as desired. For more information see Shorack and Wellner, *Empirical Processes*, 1986.

REFERENCES

[1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf