

ECE 534 FA26 HW2 Solutions to Odd-Number Problems

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The solutions to even-number problems can be found in [1, Chapter 12].

Problem 2.9 Convergence of a random sequence.

Solution.

- (a) The sequence $X_n(\omega)$ is monotone nondecreasing in n for each ω . Also, by induction on n , $0 \leq X_n(\omega) \leq 1$ for all n and ω . Since bounded monotone sequences have finite limits, $\lim_{n \rightarrow \infty} X_n$ exists in the a.s. sense and the limit is less than or equal to one with probability one.
- (b) Since a.s. convergence implies convergence in probability (see [1, Proposition 2.7 (a)]), from Part (a) we know that X_n converges in probability as well. Furthermore, we have established that $|X_n| \leq 1$ with probability one for all n . Therefore, we can apply [1, Proposition 2.7 (c)] with $Y = 1$ to deduce that $\lim_{n \rightarrow \infty} X_n$ also exists in the m.s. sense.
- (c) Since (X_n) converges a.s., it also converges in probability to the same random variable, so $Z = \lim_{n \rightarrow \infty} X_n$ a.s. It can be shown that $P\{Z = 1\} = 1$. Here is one of several proofs. Let $0 < \varepsilon < 1$. Define $a_0 := 0$, and for $k \geq 1$ define

$$a_k := \frac{a_{k-1} + 1 - \varepsilon}{2}.$$

By induction, we have $a_k = (1 - \varepsilon)(1 - 2^{-k})$. Now define for each $i \geq 1$ the event $A_i := \{U_i \geq 1 - \varepsilon\}$. These events $\{A_i\}_{i \geq 1}$ are independent, and each has probability ε . Therefore, for each $k \geq 1$, the probability that at least k of these events happens is one. Or to be more precise, let B_k be the event that at least k of the events among $\{A_i\}_{i \geq 1}$ happen, and then the probability of B_k is one¹. Now note that if at least k of these events $\{A_i\}_{i \geq 1}$ happen, then $Z \geq a_k$. Thus, we have

$$\begin{aligned} P\{(1 - \varepsilon)(1 - 2^{-k}) \leq Z \leq 1\} &\geq P(B_k) \\ &= 1, \end{aligned}$$

which implies that

$$P\{(1 - \varepsilon)(1 - 2^{-k}) \leq Z \leq 1\} = 1. \quad (1)$$

Since ε can be arbitrarily small and k can be arbitrarily large in (1), we can already expect that $P\{Z = 1\} = 1$. The following derivation shows that this is indeed true: Pick any sequence $(\varepsilon_\ell)_{\ell \geq 1}$ of numbers in $(0, 1)$ that decreases to 0 (say $\varepsilon_\ell = 2^{-\ell}$). Then we have

$$\{Z = 1\} = \bigcap_{\ell=1}^{\infty} \bigcap_{k=1}^{\infty} \{(1 - \varepsilon_\ell)(1 - 2^{-k}) \leq Z \leq 1\}. \quad (2)$$

Since the intersection of countably many probability-one events also happens with probability one (which can be shown by taking the complement and then using the union bound), we have from (1) and (2) that $P\{Z = 1\} = 1$.

Another approach: We first calculate that $E[X_n | X_{n-1} = v] = v + \frac{(1-v)^2}{4}$. Thus,

$$\begin{aligned} E[X_n] &= E[X_{n-1}] + \frac{E[(1 - X_{n-1})^2]}{4} \\ &\geq E[X_{n-1}] + \frac{(1 - E[X_{n-1}])^2}{4}, \end{aligned}$$

¹This is a direct consequence of the second Borel-Cantelli lemma. Alternatively, one can express $B_k = \cup_{m \geq 1} C_{k,m}$, where $C_{k,m}$ denotes the event where at least k events among $\{A_i\}_{1 \leq i \leq m}$ happen. A direct calculation shows that $P(C_{k,m}) = 1 - \sum_{t=0}^{k-1} \binom{m}{t} \varepsilon^t (1 - \varepsilon)^{m-t}$, which goes to 1 as m goes to infinity.

where the inequality followed from the fact that $E[W^2] \geq (E[W])^2$ for any random variable W . Since $E[X_n] \rightarrow E[Z]$ (by, e.g., Part (b) and [1, Corollary 2.13]), it follows that

$$E[Z] \geq E[Z] + \frac{(1 - E[Z])^2}{4}.$$

So $E[Z] = 1$. In view of the fact $P\{Z \leq 1\} = 1$, it follows that $P\{Z = 1\} = 1$.

2.11 Convergence of some sequences of random variables

(a) For each fixed ω ,

$$\frac{V(\omega)}{n} \rightarrow 0,$$

so $X_n(\omega) \rightarrow 1$. Thus, $X_n \rightarrow 1$ in the a.s. sense, and hence also in the p. and d. senses. Since the random variables X_n are uniformly bounded (specifically, $|X_n| \leq 1$ for all n), the convergence in p. sense implies convergence in m.s. sense as well. So $X_n \rightarrow 1$ in all four senses.

(b) Let us show Y_n does not converge in d. sense. Let F_n denote the CDF of Y_n . For any $c > 0$, we have

$$\begin{aligned} F_n(c) &= P\{Y_n \leq c\} \\ &= P\{V \leq \sqrt[n]{cn}\} \\ &= 1 - e^{-3 \sqrt[n]{cn}}. \end{aligned}$$

Therefore, we have for any $c > 0$ that

$$\begin{aligned} \lim_{n \rightarrow \infty} F_n(c) &= \lim_{n \rightarrow \infty} 1 - e^{-3 \sqrt[n]{cn}} \\ &= 1 - e^{-3}. \end{aligned}$$

Now assume for contradiction that Y_n converges to some finite random variable Y . By the definition of convergence in distribution, the CDF of Y satisfies $F_Y(c) = 1 - e^{-3}$ for all positive continuity points c of F_Y . Since there can be at most countably many discontinuity points of any CDF (a good exercise for you), we have $F_Y(c) = 1 - e^{-3}$ for all but at most countably many positive c , and then by the right continuity of F_Y we know $F_Y(c) = 1 - e^{-3}$ for all $c > 0$. But now it can be seen that $F_Y(c)$ is not a valid CDF, since it does not go to 1 as c goes to infinity. We thus have reached a contradiction. Therefore, (Y_n) does not converge in the d. sense (to a finite limit random variable), and hence does not converge in any of the four senses to a finite limit random variable.

Remark. If the question asks only about a.s. convergence, then here is a quick proof that Y_n does not converge a.s. to a finite random variable. Note that

$$P\{V \geq 0\} = 1 \quad \text{with} \quad P\{V > 1\} = e^{-3} > 0.$$

For any ω such that $V(\omega) < 1$, $Y_n(\omega) \rightarrow 0$, and for any ω such that $V(\omega) > 1$,

$$Y_n(\omega) \rightarrow +\infty,$$

so (Y_n) does not converge in the a.s. sense to a finite random variable.

(c) For each ω fixed,

$$Z_n(\omega) \rightarrow e^{V(\omega)}.$$

So $Z_n \rightarrow e^V$ in the a.s. sense, and hence also in the p. and d. senses. Using the inequality

$$1 + u \leq e^u$$

shows that $Z_n \leq e^V$ for all n , so that

$$|Z_n| \leq e^V \quad \text{for all } n.$$

Note that

$$E[(e^V)^2] = E[e^{2V}] = \int_0^\infty e^{2u} 3e^{-3u} du = 3 < \infty.$$

Therefore, the sequence (Z_n) is dominated by a single random variable with finite second moment (namely, e^V), so the convergence of (Z_n) in the p. sense to e^V implies that (Z_n) converges to e^V in the m.s. sense as well (see [1, Proposition 2.7 (c)]). So

$$Z_n \rightarrow e^V$$

in all four senses.

2.13 On the maximum of a random walk with negative drift

Solution.

- (a) By the strong law of large numbers, we have $P(A) = 1$, where A denotes the event $\{\lim_{n \rightarrow \infty} S_n/n = -1\}$. In particular, for any $\omega \in A$, there exists a positive integer $N(\omega)$ such that for all $n \geq N(\omega)$ we have

$$\frac{S_n(\omega)}{n} \leq -\frac{1}{2}.$$

It follows that $S_n(\omega) \leq -\frac{n}{2} \leq 0$ for all $n \geq N(\omega)$. Thus,

$$Z(\omega) = \max\{0, S_1(\omega), \dots, S_{N(\omega)-1}(\omega)\}$$

is a maximum over finitely many numbers, which is finite. Therefore, the event that Z is finite contains every ω in A , and thus we have $P\{Z \text{ is finite}\} \geq P(A) = 1$. It follows that $P\{Z \text{ is finite}\} = 1$.

- (b) There is no such constant L based only on the assumptions in this problem. As the hint suggests, it suffices to find a family of distributions of X_1 such that $E[X_1] = -1$ while $E[\max\{0, X_1\}]$ can be made arbitrarily large. Suppose

$$P\{X_1 = c - 1\} = P\{X_1 = -c - 1\} = 0.5$$

for a constant $c > 0$. Then X_1 has mean -1 as required. Following the hint, for $c \geq 1$, we have

$$E[Z] \geq E[\max\{0, X_1\}] = \frac{c-1}{2}.$$

Observe that $E[Z]$ can be made arbitrarily large by taking c arbitrarily large. So the answer to the question is no.

REFERENCES

- [1] B. Hajek, *Random Processes for Engineers*. Cambridge university press, 2015. [Online]. Available: <https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf>