ECE 534 FA26 HW1 Solutions to Odd-Number Problems

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The solutions to even-number problems can be found in Chapter 12 of Prof. Hajek's lecture notes.

Problem 1.3 Ordering of three random variables

Solution. For each $u \in [0,1]$, by the independence between X and Y, we have

$$\begin{aligned} P\{X < u < Y\} &= P\{X < u \text{ and } u < Y\} \\ &= P\{X < u\}P\{u < Y\} \\ &= (1 - e^{-\lambda u})e^{-\lambda u} \\ &= e^{-\lambda u} - e^{-2\lambda u}. \end{aligned}$$

Since U is uniform over $[0,1]$ and independent of X and Y, averaging over the choices of u using the PDF of U yields

$$P\{X < U < Y\} = \int_0^1 \left(e^{-\lambda u} - e^{-2\lambda u}\right) du = \frac{0.5 - e^{-\lambda} + 0.5e^{-2\lambda}}{\lambda}.$$

1.15 Poisson and geometric random variables with conditioning

Solution.

(a) First, note that

$$\begin{aligned} P\{Y < Z\} &= \sum_{i=0}^{\infty} P\{Y = i \text{ and } Z > i\} \\ &= \sum_{i=0}^{\infty} P\{Y = i\}P\{Z > i\} \\ &= \sum_{i=0}^{\infty} \frac{e^{-\mu}\mu^i}{i!} P\{Z > i\}, \end{aligned}$$

where the second equality follows from the independence between Y and Z. Then, note that for each $i \geq 0$ we have

$$P\{Z > i\} = \sum_{j=i+1}^{\infty} p(1-p)^{j-1} = (1-p)^i.$$

Therefore, we have

$$\begin{aligned} P\{Y < Z\} &= \sum_{i=0}^{\infty} \frac{e^{-\mu}\mu^i}{i!}(1-p)^i \\ &= e^{-\mu} \sum_{i=0}^{\infty} \frac{[\mu(1-p)]^i}{i!} \\ &= e^{-\mu}e^{\mu(1-p)} \\ &= e^{-\mu p}. \end{aligned}$$

(b) We have

$$
\begin{aligned}
P(Y < Z \mid Z = i) &= P(Y < i \mid Z = i) \\
&= P\{Y < i\} \\
&= \sum_{j=0}^{i-1} \frac{e^{-\mu}\mu^j}{j!},
\end{aligned}
$$

where the second equality follows again from the independence between Y and Z.

(c) We have

$$
\begin{aligned}
P(Y = i \mid Y < Z) &= \frac{P\{Y = i \text{ and } Y < Z\}}{P\{Y < Z\}} \\
&= \frac{P\{Y = i \text{ and } Z > i\}}{e^{-\mu p}} \\
&= e^{\mu p} P\{Y = i\} P\{Z > i\},
\end{aligned}
$$

where we used the fact that $P\{Y < Z\} = e^{-\mu p}$ from Part (a). Then, by the calculation in Part (a) we know that $P\{Z > i\} = (1-p)^i$. Therefore, we have

$$
\begin{aligned}
P(Y = i \mid Y < Z) &= e^{\mu p} \frac{e^{-\mu}\mu^i}{i!}(1-p)^i \\
&= \frac{e^{-\mu(1-p)}[\mu(1-p)]^i}{i!},
\end{aligned}
$$

which is the PMF of a Poisson random variable with mean $\mu(1-p)$.

(d) From Part (c), we know that the conditional distribution of Y given $Y < Z$ is the Poisson distribution with mean $\mu(1-p)$. This implies

$$
E[Y|Y < Z] = \mu(1-p).
$$