University of Illinois Fall 2013

ECE 534: Quiz

Monday September 16, 2013
7:00 p.m. — 8:00 p.m.
103 Talbot Laboratory

1. (a) Let 110 denote the outcome $X_1 = 1, X_2 = 1, X_3 = 0$, etc. Then $P\{S = 2\} = P\{110, 101, 011\} = \frac{1}{4}\frac{1}{2}\frac{1}{4} + \frac{1}{4}\frac{1}{2}\frac{3}{4} + \frac{3}{4}\frac{1}{2}\frac{3}{4} = \frac{1+3+9}{32} = \frac{13}{32}$.
   (b) As seen in part (a), there are three ways $S = 2$ can happen. The event $\{X_1 = 1\}$ happens for the first two ways. So $P(X_1 = 1|S = 2) = \frac{1+3}{13} = \frac{4}{13}$.
   (c) $\text{Cov}(X_1, S) = \text{Cov}(X_1, X_1) + \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3) = \text{Var}(X_1) = \frac{1}{4}\frac{3}{4} = \frac{3}{16}$.

2. (a) The pdf is symmetric about $\frac{\pi}{2}$, so $E[X] = \frac{\pi}{2}$.
   (b) The support of the pdf of X^2 is the interval $[0, \pi^2]$. For $0 \le c \le \pi^2$,

$$F_X(c) = P\{X^2 \le c\} = P\{X \le c^{1/2}\} = \int_0^{c^{1/2}} \frac{\sin(u)}{2} du$$

Differentiating, and taking into account the range of X, yields

$$f_{X^2}(u) = \begin{cases} \frac{\sin(\sqrt{c})}{4\sqrt{c}} & 0 \le c \le \pi^2 \\ 0 & \text{else.} \end{cases}$$

   (c) $E[\sin(X)] = \int_0^\pi \sin(u) f_X(u) du = \int_0^\pi \frac{(\sin(u))^2}{2} du = \frac{\pi}{4}$.

3. (a)

$$\begin{aligned} \text{Var}(X+Y+Z) &= \text{Cov}(X+Y+Z, X+Y+Z) \\ &= \text{Var}(X) + \text{Var}(X) + \text{Var}(X) + 2\text{Cov}(X,Y) + 2\text{Cov}(X,Z) + 2\text{Cov}(Y,Z) \\ &= 3 \cdot 4 + 6 \cdot 1 = 18 \end{aligned}$$

   (b) Need θ so $\text{Cov}(X + \theta Y, Z) = 0$, or $1 + \theta = 0$. So $\theta = -1$.

4. (a)

$$E[XY] = \int_0^1 \int_0^1 u(3u^2) v dv du = \frac{1}{4} \cdot 3 \cdot \frac{1}{2} = \frac{3}{8}.$$

   (b)

$$f_Y(v) = \begin{cases} \int_0^1 3u^2 \, du = 1 & 0 \le v \le 1 \\ 0, & \text{else} \end{cases}$$

That is, Y has the uniform distribution over the interval $[0, 1]$. This could have been deduced by inspection, using the fact the random variables are independent.