University of Illinois Fall 2013

ECE 534 Final Exam

Tuesday, December 17, 2013
1:30 p.m. — 4:30 p.m.
106B1&B8 Engineering Hall

1. (a) Since $R_X(0) = 5$, $R_X(1) = 0$, and $R_X(2) = -\frac{5}{9}$, the covariance matrix of the vector is $\begin{pmatrix} 5 & 0 & -5/9 \\ 0 & 5 & 0 \\ -5/9 & 0 & 5 \end{pmatrix}$.

   (b) The variables are jointly Gaussian so the expectation is the best linear estimator. The variables $X(2)$ and $X(3)$ are uncorrelated (so independent) and the variables are mean zero, so

$$E[X(4)|X(2), X(3)] = \frac{\text{Cov}(X(4), X(2))X(2)}{\text{Var}(X(2))} + \frac{\text{Cov}(X(4), X(3))X(3)}{\text{Var}(X(2))} = -\frac{X(2)}{9}.$$

2. (a) Let $P_k = E[X_k^2]$. Note that $A_k X_k$ is orthogonal to B_k, and A_k is independent of X_k. Thus $P_{k+1} = E[(A_k X_k)^2] + E[B_k^2] = \sigma_A^2 P_k + \sigma_B^2$. (Some students solved for the X_k's in terms of the A_k's and B_k's and then took expectations to get an expression for P_k. That answer was accepted.)

   (b) Yes, the sequence X_k is determined by a recursion driven by a sequence of independent random vectors, $((A_k, B_k) : k \geq 0)$. The proof of the Markov property in this case is identical to the case that a sequence is determined by a recursion driven by a sequence of independent random variables.

   (c) The random variables X_k are already orthogonal, so the innovations sequence is simply the same as the original sequence.

3. (a) $Q = \begin{pmatrix} -a-1 & a & 1 \\ 1 & -b-1 & b \\ c & 1 & -c-1 \end{pmatrix}$ and simple algebra shows that $(1 + c + cb, 1 + a + ac, 1 + b + ba)Q = (0, 0, 0)$. (Since the row sums of Q are zero it suffices to check two of the equations. By symmetry in fact it suffices to check just the first equation.

   (b) The long term rate of jumps from state 1 to state 2 is $\pi_1 a$ and the long term rate of jumps from state 2 to 1 is π_2. The difference is the mean cycle rate: $\theta = \pi_1 a - \pi_2$. Similarly, $\theta = \pi_2 b - \pi_3$ and $\theta = \pi_3 c - \pi_1$.
   ALTERNATIVELY, the average rate of clockwise jumps per unit time is $\pi_1 a + \pi_2 b + \pi_3 c$ and the average rate of counterclockwise jumps is one. So the net rate of jumps in the clockwise direction is $\pi_1 a + \pi_2 b + \pi_3 c - 1$. Since there are three jumps to a cycle, divide by three to get $\theta = (\pi_1 a + \pi_2 b + \pi_3 c - 1)/3$.

   (c) By part (a), $\pi = (1+c+cb, 1+a+ac, 1+b+ba)/Z$ where $Z = 3+a+b+c+ab+ac+bc$. So then using part (b), $\theta = \frac{(1+c+bc)a-1-a-ac}{Z} = \frac{abc-1}{3+a+b+c+ab+ac+bc}$. The mean net cycle rate is zero if and only if $abc = 1$. (Note: The nice form of the equilibrium for this problem, which generalizes to rings of any integer circumference, is a special case of the tree based formula for equilibrium distributions that can be found, for example, in the book of Freidlin and Wentzell, *Random perturbations of dynamical systems.*

4. (a) $P(X = i|Y_1 = 1, Y_2 = 2, Y_3 = 2) = \frac{P(X=i,Y_1=1,Y_2=2,Y_3=2)}{P(Y_1=1,Y_2=2,Y_3=2)} = \frac{a_{i,1}a_{i,2}a_{i,2}}{\sum_{i'} a_{i',1}a_{i',2}a_{i',2}}$ so $\widehat{X}_{MAP} = \arg\max_i a_{i,1}a_{i,2}a_{i,2} = 2$. Here we use the fact that $a_{i,1}a_{i,2}a_{i,2}$ is 2 for $i = 1$, 4 for $i = 2$, and 1 for $i = 3$.

(b) By the calculations in part (a),

$$P(X=i|Y_1=1,Y_2=2,Y_3=2)=\begin{cases}\frac{2}{7} & i=1\\ \frac{4}{7} & i=2\\ \frac{2}{7} & i=3\end{cases}$$

5. (a) Fix h and let $Y_t = X_{t+h}X_t$. Clearly Y is stationary with mean $\mu_Y = R_X(h)$. Observe that

$$\begin{aligned} C_Y(\tau) &= E[Y_\tau Y_0] - \mu_Y^2 \\ &= E[X_{\tau+h}X_\tau X_h X_0] - R_X(h)^2 \\ &= R_X(h)^2 + R_X(\tau)R_X(\tau) + R_X(\tau+h)R_X(\tau-h) - R_X(h)^2 \\ &= R_X(\tau)R_X(\tau) + R_X(\tau+h)R_X(\tau-h) \end{aligned}$$

Therefore, $C_Y(\tau)\to 0$ as $|\tau|\to\infty$. Hence Y is mean ergodic, so X is correlation ergodic. (Note that $R_Y(\tau)\to R_X(0)^2$ as $\tau\to\infty$.)

(b) $X_t = A\cos(t+\Theta)$, where A is a random variable with positive variance, Θ is uniformly distributed on the interval $[0,2\pi]$, and A is independent of Θ. Note that $\mu_X = 0$ because $E[\cos(t+\Theta)] = 0$. Also, $|\int_0^T X_t dt| = |A\int_0^T \cos(t+\Theta)dt| \le 2|A|$ so $\left|\frac{\int_0^T X_t dt}{T}\right| \le \frac{2|A|}{T} \to 0$ in the m.s. sense. So X is m.s. ergodic. Similarly, we have $\frac{\int_0^T X_t^2 dt}{T} \to \frac{A^2}{2}$ in the m.s. sense. The limit is random, so X_t^2 is not mean ergodic, so X is not correlation ergodic. (The definition is violated for $h=0$.)
ALTERNATIVELY $X_t = \cos(Vt+\Theta)$ where V is a positive random variable with nonzero variance, Θ is uniformly distributed on the interval $[0,2\pi]$, and V is independent of Θ. In this case, X is correlation ergodic as before. But $\int_0^T X_t X_{t+h}dt \to \frac{\cos(Vh)}{2}$ in the m.s. sense. This limit is random, at least for some values of h, so Y is not mean ergodic so X is not correlation ergodic.

6. (a)

$$E\left[\int_0^T X_t^2 dt\right] = E[B^2]\int_0^T \sin^2\left(\frac{\pi t}{T}\right)dt = \int_0^T \frac{1-\cos(\frac{2\pi t}{T})}{2}dt = \frac{T}{2}$$

(b) The definition of X is equivalent to $X_t = (B\sqrt{T/2})\phi_1(t)$ where $\phi_1(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{\pi t}{T}\right)$, which is a KL expansion of X with a single nonzero term. Also, $\lambda_1 = E\left[\left(B\sqrt{T/2}\right)^2\right] = \frac{T}{2}$. Alternatively, since there is only one (nonzero) term in the KL expansion, λ_1 is equal to the total average energy found in part (a).

(c) The facts $X_1 = B\sqrt{T/2}$, N_1 has mean zero and variance σ^2, and $Y_1 = X_1 + N_1$ imply

$$E[B|Y_1] = \frac{\mathrm{Cov}(B,Y_1)Y_1}{\mathrm{Var}(Y_1)} = \frac{(\sqrt{\frac{T}{2}})Y_1}{\sigma^2+\frac{T}{2}}$$

7. (a) By the orthogonality principle, it suffices for $X_T - \widehat{X}_T \perp X_u$ for all $u \le 0$. Or $E\left[\left(X_T - \int_0^T g(t)X_t dt\right)X_u\right] = 0$ for all $u\le 0$. Which yields the desired equations: $R_X(T-u) = \int_{-\infty}^0 g(t)R_X(t-u)dt$ for all $u \le 0$.

(b) The Markov property of X implies that $\widehat{X}_T = E[X_T|X_0] = \frac{R_X(T)X_0}{R_X(0)} = X_0 e^{-\alpha T}$. Equivalently, $g(t) = \delta(t)e^T$, which we see satisfies the conditions of part (a).

8. (a) Yes, the integral of a m.s. continuous process is (continuously) m.s. differentiable.

(b) No, R_Y is not even continuous, so Y is not even m.s. continuous, so is not m.s. differentiable.

(c) Yes. We should expect the answer is yes because if the sum is differentiated term by term, the resulting sum is still m.s. convergent. That is, we expect $Z'_t = Y_t \stackrel{\triangle}{=} \sum_{n=0}^{\infty} \frac{V_n \cos(nt)}{n}$. To check this out, note that

$$E\left[\left(\frac{Z_{t+h}-Z_t}{h} - Y_t\right)^2\right] = E\left[\left(\sum_{n=1}^{\infty} V_n \left(\frac{\sin(n(t+h)) - \sin(nt)}{n^2 h} - \frac{\cos(nt)}{n}\right)\right)^2\right]$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{\sin(n(t+h)) - \sin(nt)}{nh} - \cos(nt)\right)^2$$

For each $n \geq 1$, $\frac{\sin(n(t+h))-\sin(nt)}{nh} - \cos(nt) \to 0$ as $h \to 0$ and the terms $\frac{\sin(n(t+h))-\sin(nt)}{nh} - \cos(nt)$ are bounded by 2 (use the intermediate value form of Taylor's theorem). Hence by the dominated convergence theorem,
$\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{\sin(n(t+h))-\sin(nt)}{nh} - \cos(nt)\right)^2 \to 0$ as $h \to 0$.
ALTERNATIVE, note that, since the terms in the series defining Z are orthogonal random variables, $R_Z(s,t) = \sum_{n=1}^{\infty} \frac{\sin(ns)\sin(nt)}{n^4}$. From this we see that $\partial_2 R_Z$ and $\partial_1 \partial_2 R_Z$ exist and are continuous. Therefore, R_Z is continually m.s. differentiable.