**University of Illinois** Fall 2013

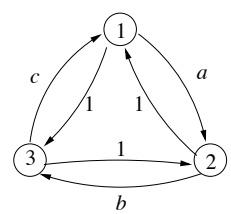
# ECE 534 Final Exam

Tuesday, December 17, 2013
1:30 p.m. — 4:30 p.m.
106B1&B8 Engineering Hall

1. **[20 points]** Let $X$ be a stationary Gaussian process with $\mu_X = 0$ and $R_X(\tau) = 5\cos(\pi\tau/2)3^{-|\tau|}$.
   (a) Find the covariance matrix of the random vector $(X(2), X(3), X(4))^T$.
   (b) Find $E[X(4)|X(2), X(3)]$.

2. **[30 points]** Let all the variables $A_k, B_k,\ k \geq 0$ be mutually independent with mean zero. Let $E[A_k^2] = \sigma_A^2$ and $E[B_k^2] = \sigma_B^2$ for all $k \geq 0$, for some positive constants $\sigma_A^2$ and $\sigma_B^2$. Finally, define the sequence $(X_k : k \geq 0)$ by $X_0 = 0$ and $X_{k+1} = A_k X_k + B_k$ for $k \geq 0$.
   (a) Find a recursive method for computing $E[X_k^2]$ for $k \geq 0$.
   (b) Is $(X_k : k \geq 0)$ a Markov sequence? Justify your answer.
   (c) Find the corresponding innovations sequence $\tilde{X}_k$.

3. **[30 points]** Consider a continuous time Markov process with the transition rate diagram shown, where $a, b$, and $c$ are strictly positive constants.




   (a) Write down the $Q$ matrix and verify that the equilibrium probability distribution $\pi$ is proportional to $(1 + c + cb, 1 + a + ac, 1 + b + ba)$.
   (b) Depending on the values of $a, b$ and $c$, the process may tend to cycle clockwise, cycle counter clockwise, or tend to be cycle neutral. For example, it is cycle neutral if $a = b = c = 1$. Let $\theta$ denote the long term rate of cycles per second in the clockwise direction per unit time. (A negative value indicates a long term rate of rotation in the counter clockwise direction.) For example, if $a = b = c$ then $\theta = (a - 1)/3$. Give a simple expression for $\theta$ in terms of $\pi$, $a, b$, and $c$.
   (c) Express $\theta$ in terms of $a, b$, and $c$. What condition on $a, b$ and $c$ is equivalent to the mean net cycle rate being zero?

4. **[20 points]** Let $A$ be the three by three matrix with entries $a_{i,j} = \begin{cases} 2 & i = j \\ 1 & i \neq j \end{cases}$. Suppose $X, Y_1, Y_2, Y_3$ have the joint pmf $P\{X = i, Y_1 = j, Y_2 = k, Y_3 = l\} = \frac{a_{i,j}a_{i,k}a_{i,l}}{Z}$, where $Z$ is a normalizing constant so that the sum of $P\{X = i, Y_1 = j, Y_2 = k, Y_3 = l\}$ over all $i, j, k, l \in \{1, 2, 3\}$ is equal to one.
   (a) Find the maximum a posteriori (MAP) estimate of $X$ given $(Y_1, Y_2, Y_3) = 122$.
   (b) Find the conditional probability distribution of $X$ given $(Y_1, Y_2, Y_3) = 122$.

5. **[20 points]** A real valued WSS random process $X$ is called *correlation ergodic* (in the m.s. sense) if for any constant $h$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_{s+h}X_s ds = E[X_{s+h}X_s] \quad (m.s. \text{ sense})$$

   (a) Suppose $X$ is a mean zero, real-valued Gaussian process such that $R_X(\tau) \to 0$ as $|\tau| \to \infty$. Show that $X$ is correlation ergodic. (Hints: Let $Y_t = X_{t+h}X_t$. Then correlation ergodicity of $X$ is equivalent to mean ergodicity of $Y$. If $A, B, C$, and $D$ are mean zero, jointly Gaussian random variables, then $E[ABCD] = E[AB]E[CD] + E[AC]E[BD] + E[AD]E[BC]$.)

   (b) Give a simple example of a WSS random process that is mean ergodic in the m.s. sense but is not correlation ergodic in the m.s. sense.

6. **[30 points]** Let $X = (X_t : 0 \le t \le T)$ be a random process given by $X_t = B\sin(\frac{\pi t}{T})$, where $T$ is a positive constant and $B$ is a $N(0,1)$ random variable.

   (a) What is the expected total energy of $X$?

   (b) Describe the Karhunen-Loéve expansion of $X$. (You don't need to explicitly identify the eigenfunctions for eigenvalues that are zero.)

   (c) Let $N = (N_t : 0 \le t \le T)$ be a real-valued Gaussian white noise process independent of $X$ with $R_N(\tau) = \sigma^2\delta(\tau)$, and let $Y = X + N$. Think of $Y$ as a noisy observation of $X$. The same basis functions used for $X$ can be used for the Karhunen-Loève expansions of $N$ and $Y$. Let $X_1 = \langle X, \phi_1\rangle$, $N_1 = \langle N, \phi_1\rangle$, and $Y_1 = \langle Y, \phi_1\rangle$. Note that $Y_1 = X_1 + N_1$. Find $E[B|Y_1]$. (Remark: The other coordinates $Y_2, Y_3, \ldots$ are independent of both $X$ and $Y_1$, and are thus useless for the purpose of estimating $B$. Thus, $E[B|Y_1]$ is equal to $E[B|Y]$, the MMSE estimate of $B$ given the entire observation process $Y$.)

7. **[20 points]** Let $X$ be a Gaussian stationary process with $R_X(\tau) = e^{-|\tau|}$ and mean zero. Suppose $X_T$ is to be estimated given $(X_t : t \le 0)$ where $T$ is a fixed positive constant, and the mean square error is to be minimized. Without loss of generality, suppose the estimator has the form $\widehat{X}_T = \int_0^T g(t)X_t dt$ for some (possibly generalized) function $g$.

   (a) Using the orthogonality principle, find equations that characterize $g$.

   (b) Identify the solution $g$. (Hint: Does $X$ have any special properties?)

8. **[30 points]** For each process described below, determine whether the process is m.s. differentiable in the m.s. sense. Justify your reasoning.

   (a) $X_t = \int_0^t N_s ds$, where $N$ is a Poisson random process with rate parameter one.

   (b) Process $Y$, assumed to be a mean-zero Gaussian process with autocorrelation function $R_Y(s,t) = \begin{cases} 1 & \text{if } \lfloor s\rfloor = \lfloor t\rfloor \\ 0 & \text{else.} \end{cases}$ Here "$\lfloor x\rfloor$" denotes the greatest integer less than or equal to $x$.

   (c) Process $Z$ defined by the series (which converges uniformly in the m.s. sense) $Z_t = \sum_{n=1}^{\infty} \frac{V_n \sin(nt)}{n^2}$ where the $V_n$'s are independent, $N(0,1)$ random variables.