University of Illinois Fall 2013

ECE 534 Exam 1

Monday October 14, 2013
7:00 p.m. — 8:15 p.m.
124 Burrill Hall

1. (a) For each ω fixed, $\frac{V(\omega)}{n} \to 0$ so $X_n(\omega) \to 1$. Thus, $X_n \to 1$ in the a.s sense, and hence also in the p. and d. senses. Since the random variables X_n are uniformly bounded (specifically, $|X_n| \leq 1$ for all n), the convergence in p. sense implies convergence in m.s. sense as well. So $X_n \to 1$ in all four senses.

   (b) To begin we note that $P\{V \geq 0\} = 1$ with $P\{V > 1\} = e^{-3} > 0$. For any ω such that $V(\omega) < 1$, $Y_n(\omega) \to 0$, and for any ω such that $V(\omega) > 1$, $Y_n(\omega) \to +\infty$, so (Y_n) does not converge in the a.s. sense to a finite random variable.

   Let us try to show Y_n does not converge in d. sense. For any $c > 0$ $\lim_{n\to\infty} F_n(c) = \lim_{n\to\infty} P\{Y_n \leq c\} = P\{V < 1\} = 1 - e^{-3}$. The limit exists but the limit function F satisfies $F(c) = e^{-1}$ for all $c > 0$, so the limit is not a valid CDF. Thus, (Y_n) does not converge in the d. sense (to a finite limit random variable), and hence does not converge in any of the four senses to a finite limit random variable.

   (c) For each ω fixed, $Z_n(\omega) \to e^{V(\omega)}$. So $Z_n \to e^V$ in the a.s. sense, and hence also in the p. and d. senses. Using the inequality $1 + u \leq e^u$ shows that $Z_n \leq e^V$ for all n so that $|Z_n| \leq e^V$ for all n. Note that $E[(e^V)^2] = E[e^{2V}] = \int_0^\infty e^{2u} 3e^{-3u} du = 3 < \infty$. Therefore, the sequence (Z_n) is dominated by a single random variable with finite second moment (namely, e^V), so the convergence of (Z_n) in the p. sense to e^V implies that (Z_n) converges to e^V in the m.s. sense as well. So $Z_n \to e^V$ in all four senses.

2. (a) The required bound is provided by Chernoff's inequality for any $c > 0.5$ because the U's have mean 0.5. If $c = 0.5$ the probability is exactly 0.5 for all n and doesn't satisfy the required bound for any $b > 0$. Hence $c < 0.5$ doesn't work either. In summary, the bound holds precisely when $c > 0.5$.

   (b) The probability in question is equal to $P\{X_1 + \cdots + X_n > 0\}$, where $X_k = U_k - cU_{n+k}$ for $1 \leq k \leq n$. The X's are iid and $E[X_k] = \frac{1-c}{2}$. So if $c < 1$ the required bound is provided by Chernoff's inequality applied to the X's. If $c = 1$ the probability is exactly 0.5 for all n and doesn't satisfy the required bound for any $b > 0$. Hence $c > 1$ doesn't work either. In summary, such $b > 1$ exists if and only if $c < 1$.

3. (a) True. The minimum MSE over all functions of Y is less than or equal to the minimum MSE over all functions of Y of the form $a + bY + cY^2$.

   (b) False. Suppose X has an even distribution (i.e. X and $-X$ have the same distribution) such that X is not always zero, and let $Y = X$. Then $E[X|Y^2] = 0$ while $\widehat{E}[X|Y] = X$, so the inequality becomes $E[X^2] \leq 0$, which is a contradiction.

   (c) True. The statement means $E[X]$ is the function of Y with the MMSE for estimating X. By the orthogonality principle, $X - E[X]$ is thus orthogonal to any (square integrable) function of Y, including Y itself. So $0 = E[(X - E[X])Y] = \mathrm{Cov}(X, Y)$.

4. (a) $\widetilde{Y}_1 = Y_1$ and $\widetilde{Y}_2 = Y_1 - \frac{\mathrm{Cov}(Y_2,Y_1)Y_1}{\mathrm{Var}(Y_1)} = Y_2 - (0.5)Y_1$.

   (b) Using the answer to part (a) and the fact projections onto orthogonal random variables reduce to sums of projections, yields (this is somewhat simplified by the fact the means are zero):

$$\begin{aligned}
\widehat{E}[X|Y_1,Y_2] &= \overline{X} + \widehat{E}[X-\overline{X}|Y_1] + \widehat{E}[X-\overline{X}|\widetilde{Y}_2] \\
&= \widehat{E}[X|Y_1] + \widehat{E}[X|\widetilde{Y}_2] \\
&= \frac{\mathrm{Cov}(X,Y_1)Y_1}{\mathrm{Var}(Y_1)} + \frac{\mathrm{Cov}(X,\widetilde{Y}_2)\widetilde{Y}_2}{\mathrm{Var}(\widetilde{Y}_2)} \\
&= \frac{(0.5)Y_1}{1} + \frac{(0-0.25)(Y_2-(0.5)Y_1)}{1+(0.5)^2-2(0.5)^2} \\
&= \frac{2Y_1-Y_2}{3}
\end{aligned}$$

ALTERNATIVELY, using the formula for $\widehat{E}[X|Y_1,Y_2]$,

$$\begin{aligned}
\widehat{E}[X|Y_1,Y_2] &= \mathrm{Cov}\left(X, \begin{pmatrix}Y_1\\Y_2\end{pmatrix}\right)\left(\mathrm{Cov}\begin{pmatrix}Y_1\\Y_2\end{pmatrix}\right)^{-1}\begin{pmatrix}Y_1\\Y_2\end{pmatrix} \\
&= (0.5\ \ 0)\begin{pmatrix}1 & 0.5\\ 0.5 & 1\end{pmatrix}^{-1}\begin{pmatrix}Y_1\\Y_2\end{pmatrix} \\
&= (0.5\ \ 0)\begin{pmatrix}\frac{4}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{4}{3}\end{pmatrix}\begin{pmatrix}Y_1\\Y_2\end{pmatrix} \\
&= \frac{2Y_1-Y_2}{3}
\end{aligned}$$