ECE 534 RANDOM PROCESSES **FALL 2008**
PROBLEM SET 7 **Due Wednesday, December 10**

7. Random Processes in Linear Systems and Spectral Analysis

Assigned Reading: Sections 8.1-8.4 and 9.1 of the notes.
Reminder: The final exam will be on Monday, December 15, 1:30-4:30 p.m., Room 66 in the Main Library, which is the large building just south of Gregory Hall. Room 66 is located in the northwest basement, accessed by the west elevator. There is also outside access by stairs near the north library door.

Problems to be handed in:

1 Some linear transformations of some random processes

Let $U = (U_n : n \in \mathbb{Z})$ be a random process such that the variables U_n are independent, identically distributed, with $E[U_n] = \mu$ and $\mathrm{Var}(U_n) = \sigma^2$, where $\mu \neq 0$ and $\sigma^2 > 0$. Please keep in mind that $\mu \neq 0$. Let $X = (X_n : n \in \mathbb{Z})$ be defined by $X_n = \sum_{k=0}^{\infty} U_{n-k} a^k$, for a constant a with $0 < a < 1$.
(a) Is X stationary? Find the mean function μ_X and autocovariance function C_X for X.
(b) Is X a Markov process ? (Hint: X is not necessarily Gaussian. Does X have a state representation driven by U?)
(c) Is X mean ergodic in the m.s. sense?
Let U be as before, and let $Y = (Y_n : n \in \mathbb{Z})$ be defined by $Y_n = \sum_{k=0}^{\infty} U_{n-k} A^k$, where A is a random variable distributed on the interval $(0, 0.5)$ (the exact distribution is not specified), and A is independent of the random process U.
(d) Is Y stationary? Find the mean function μ_Y and autocovariance function C_Y for Y. (Your answer may include expectations involving A.)
(e) Is Y a Markov process? (Give a brief explanation.)
(f) Is Y mean ergodic in the m.s. sense?

2 Causal prediction of a Poisson process

Let $N = (N_t : t \geq 0)$ be a Poisson process with rate $\lambda > 0$.
(a) Let $S = \min\{t \geq 0 : N_t \geq 3\}$, which is the time of the third jump of N.Write down the probability density of S, being as explicit as possible.
(b) Find $\widehat{E}[N_{20}|N_s, 0 \leq s \leq 10]$, which is defined to be the minimum mean square error linear estimator of N_{20} given $(N_s : 0 \leq s \leq 10)$. (Hint: No calculation is necessary.)

3 On an $M/D/$infinity system

Suppose customers enter a service system according to a Poisson point process on $\mathbb{R}$ of rate λ, meaning that the number of arrivals, $N(a, b]$, in an interval $(a, b]$, has the Poisson distribution with mean $\lambda(b-a)$, and the numbers of arrivals in disjoint intervals are independent. Suppose each customer stays in the system for one unit of time, independently of other customers. Because the arrival process is memoryless, because the service times are deterministic, and because the customers are served simultaneously, corresponding to infinitely many servers,

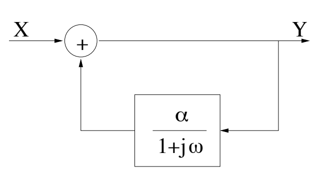
this queueing system is called an $M/D/\infty$ queueing system. The number of customers in the system at time t is given by $X_t = N(t-1, t]$.
(a) Find the mean and autocovariance function of X.
(b) Is X stationary? Is X wide sense stationary?
(c) Is X a Markov process?
(d) Find a simple expression for $P\{X_t = 0 \text{ for } t \in [0,1]\}$ in terms of λ.
(e) Find a simple expression for $P\{X_t > 0 \text{ for } t \in [0,1]\}$ in terms of λ.

4 Filtering Poisson white noise

A Poisson random process $N = (N_t : t \geq 0)$ has indpendent increments. The derivative of N, written N', does not exist as an ordinary random process, but it does exist as a generalized random process. Graphically, picture N' as a superposition of delta functions, one at each arrival time of the Poisson process. As a generalized random process, N' is stationary with mean and autocovariance functions given by $E[N'_t] = \lambda$, and $C_{N'}(s,t) = \lambda\delta(s-t)$, repectively, because, when integrated, these functions give the correct values for the mean and covariance of N: $E[N_t] = \int_0^t \lambda ds$ and $C_N(s,t) = \int_0^s \int_0^t \lambda\delta(u-v)dvdu$. The random process N' can be extended to be defined for negative times by augmenting the original random process N by another rate λ Poisson process for negative times. Then N' can be viewed as a WSS random process, and its integral over intervals gives rise to a process $N(a,b]$ as described in Problem 3. (The process $N' - \lambda$ is a white noise process, in that it is a generalized random process which is WSS, mean zero, and has autocorrelation function $\lambda\delta(\tau)$. Both N' and $N' - \lambda$ are called Poisson shot noise processes. One application for such processes is modeling noise in small electronic devices, in which effects of single electrons can be registered. For the remainder of this problem, N' is used instead of the mean zero version.) Let X be the output when N' is passed through a linear time-invariant filter with an impulse response function h, such that $\int_{-\infty}^{\infty} |h(t)|dt$ is finite. (Remark: In the special case that $h(t) = I_{\{0 \leq t < 1\}}$, X is the $M/D/\infty$ process of Problem 3.)
(a) Find the mean function and covariance functions of X.
(b) Consider the special case that $h(t) = e^{-t} I_{\{t \geq 0\}}$. Explain why X is a Markov process in this case. (Hint: What is the behavior of X between the arrival times of the Poisson process? What does X do at the arrival times?)

5 A linear system with a feedback loop

The system with input X and output Y involves feedback with the loop transfer function shown.



(a) Find the transfer function K of the system describing the mapping from X to Y.
(b) Find the corresponding impluse response function.

(c) The power of Y divided by the power of X, depends on the power spectral density, S_X. Find the supremum of this ratio, over all choices of S_X, and describe what choice of S_X achieves this supremum.

6 Linear and nonlinear reconstruction from samples

Suppose $X_t = \sum_{n=-\infty}^{\infty} g(t-n-U)B_n$, where the B_n's are independent with mean zero and variance $\sigma^2 > 0$, g is a function with finite energy $\int |g(t)|^2 dt$ and Fourier transform $G(\omega)$, U is a random variable which is independent of B and uniformly distributed on the interval $[0,1]$. The process X is a typical model for a digital baseband signal, where the B_n's are random data symbols.
(a) Show that X is WSS, with mean zero and $R_X(t) = \sigma^2 g * \widetilde{g}(t)$.
(b) Under what conditions on G and T can the sampling theorem be used to recover X from its samples of the form $(X(nT) : n \in \mathbb{Z})$?
(c) Consider the particular case $g(t) = (1-|t|)_+$ and $T = 0.5$. Although this falls outside the conditions found in part (b), show that by using nonlinear operations, the process X can be recovered from its samples of the form $(X(nT) : n \in \mathbb{Z})$. (Hint: Consider a sample path of X)

7 Estimation of a process with raised cosine spectrum

Suppose $Y = X + N$, where X and N are independent, mean zero, WSS random processes with

$$S_X(\omega) = \frac{(1+\cos(\frac{\pi\omega}{\omega_o}))}{2} I_{\{|\omega|\leq\omega_o\}} \quad \text{and} \quad S_N(\omega) = \frac{N_o}{2}$$

where $N_o > 0$ and $\omega_o > 0$. (a) Find the transfer function H for the filter such that if the input process is Y, the output process, $\widehat{X}$, is such that $\widehat{X}$ is the optimal linear estimator of X_t based on $(Y_s : s \in \mathbb{R})$.
(b) Express the mean square error, $\sigma_e^2 = E[(\widehat{X}_t - X_t)^2]$, as an integral in the frequency domain. (You needn't carry out the integration.)
(c) Describe the limits of your answers to (a) and (b) as $N_o \to 0$.
(c) Describe the limits of your answers to (a) and (b) as $N_o \to \infty$.