ECE 534 RANDOM PROCESSES **FALL 2008**
PROBLEM SET 5 **Due Wednesday, November 5**

5. Two faces of Markov processes–inference and dynamics

Assigned Reading: Sections 4.9 and 4.10, Chapter 5, and Chapter 6 through section 8.

Problems to be handed in:

1 A two station pipeline in continuous time

This is a continuous-time version of Example 4.9.1. Consider a pipeline consisting of two single-buffer stages in series. Model the system as a continuous-time Markov process. Suppose new packets are offered to the first stage according to a rate λ Poisson process. A new packet is accepted at stage one if the buffer in stage one is empty at the time of arrival. Otherwise the new packet is lost. If at a fixed time t there is a packet in stage one and no packet in stage two, then the packet is transfered during $[t, t+h)$ to stage two with probability $h\mu_1 + o(h)$. Similarly, if at time t the second stage has a packet, then the packet leaves the system during $[t, t+h)$ with probability $h\mu_2 + o(h)$, independently of the state of stage one. Finally, the probability of two or more arrival, transfer, or departure events during $[t, t+h)$ is $o(h)$. (a) What is an appropriate state-space for this model? (b) Sketch a transition rate diagram. (c) Write down the Q matrix. (d) Derive the throughput, assuming that $\lambda = \mu_1 = \mu_2 = 1$.

2 A variance estimation problem with Poisson observation

The input voltage to an optical device is X and the number of photons observed at a detector is N. Suppose X is a Gaussian random variable with mean zero and variance σ^2, and that given X, the random variable N has the Poisson distribution with mean X^2. (Recall that the Poisson distribution with mean λ has probability mass function $\lambda^n e^{-\lambda}/n!$ for $n \geq 0$.)
(a) Express $P[N = n|\sigma^2]$ as an integral. You do not have to perform the integration.
(b) Find the maximum likelihood estimator of σ^2 given N. (Caution: Estimate σ^2, not X. Be as explicit as possible–the final answer has a simple form. Hint: You can first simplify your answer to part (a) by using the fact that if X is a $N(0, \widetilde{\sigma}^2)$ random variable, then $E[X^{2n}] = \frac{\widetilde{\sigma}^{2n}(2n)!}{n!2^n}$.)

3 Finding a most likely path

Consider an HMM with state space $\mathcal{S} = \{0, 1\}$, observation space $\{0, 1, 2\}$, and parameter $\theta = (\pi, A, B)$ given by:

$$\pi = (a, a^3) \quad A = \begin{pmatrix} a & a^3 \\ a^3 & a \end{pmatrix} \quad B = \begin{pmatrix} ca & ca^2 & ca^3 \\ ca^2 & ca^3 & ca \end{pmatrix}$$

Here a and c are positive constants. Their actual numerical values aren't important, other than the fact that $a < 1$. Find the MAP state sequence for the observation sequence 021201, using the Viterbi algorithm. Show your work.

4 Specialization of Baum-Welch algorithm for no hidden data

Suppose that the matrix B is the identity matrix, so that $X_t = Y_t$ for all t. (a) Determine how the Baum-Welch algorithm simplifies in the special case that B is the identity matrix, so that $X_t = Y_t$ for all t. (b) Still assuming that B is the identity matrix, suppose that $\mathcal{S} = \{0, 1\}$ and the observation sequence is 0001110001110001110001. Find the ML estimator for π and A.

5 Baum-Welch saddlepoint

Suppose that the Baum-Welch algorithm is run on a given data set with initial parameter $\theta^{(0)} = \theta = (\pi, A, B)$ such that $\pi = \pi A$ (i.e., the initial distribution of the state is an equilibrium distribution of the state) and every row of $B^{(0)}$ is identical. Explain what happens, assuming an ideal computer with infinite precision arithmetic is used.

6 Constraining the Baum-Welch algorithm

The Baum-Welch algorithm as presented placed no prior assumptions on the parameters π, A, B, other than the number of states N_s in the state space of (Z_t). Suppose matrices $\overline{A}$ and $\overline{B}$ are given with the same dimensions as the matrices A and B to be esitmated, with all elements of $\overline{A}$ and $\overline{B}$ having values 0 and 1. Suppose that A and B are constrained to satisfy $A \leq \overline{A}$ and $B \leq \overline{B}$, in the element-by-element ordering (for example, $a_{ij} \leq \overline{a}_{ij}$ for all i, j.) Explain how the Baum-Welch algorithm can be adapted to this situation.

7 Mean hitting time for a simple Markov process

Let $(X(n) : n \geq 0)$ denote a discrete-time, time-homogeneous Markov chain with state space $\{0, 1, 2, 3\}$ and one-step transition probability matrix

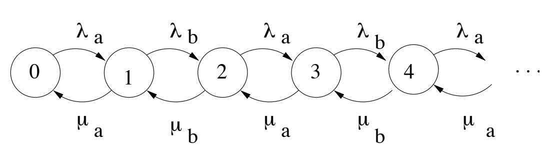
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-a & 0 & a & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for some constant a with $0 \leq a \leq 1$. (a) Sketch the transition probability diagram for X and give the equilibrium probability vector. If the equilibrium vector is not unique, describe all the equilibrium probability vectors.

(b) Compute $E[\min\{n \geq 1 : X(n) = 3\}|X(0) = 0]$.

8 A birth-death process with periodic rates

Consider a single server queueing system in which the number in the system is modeled as a continuous time birth-death process with the transition rate diagram shown, where $\lambda_a, \lambda_b, \mu_a$, and μ_b are strictly positive constants.



(a) Under what additional assumptions on these four parameters is the process positive recurrent?

(b) Assuming the system is positive recurrent, under what conditions on $\lambda_a, \lambda_b, \mu_a$, and μ_b is it true that the distribution of the number in the system at the time of a typical arrival is the same as the equilibrium distribution of the number in the system?