A non-measurable set in $(0, 1]$
6.975, Fall 2004

Let "+" stand for addition modulo 1 in $(0, 1]$. For example, $0.5 + 0.7 = 0.2$, instead of 1.2. If $A \subseteq (0, 1]$, and x is a number, then $A + x$ stands for the set of all numbers of the form $y + x$ where $y \in A$. You may want to visualize $(0, 1]$ as a circle that wraps around so that after 1, one starts again at 0.

Define x and y to be equivalent if $x + r = y$ for some rational number r. Then, $(0, 1]$ can be partitioned into equivalent classes. (That is, all elements in the same equivalence class are equivalent, elements belonging to different equivalent classes are not equivalent, and every $x \in (0, 1]$ belongs to some equivalence class.) Let us pick exactly one element from each equivalence class, and let H be the set of the elements picked this way. (This fact that a set H can be legitimately formed this way involves the Axiom of Choice, a generally accepted axiom of set theory.) We will now consider the sets of the form $H + r$, where r ranges over the rational numbers in $(0, 1]$. Note that there are countably many such sets.

The sets $H + r$ are disjoint. (Indeed, if $r_1 \neq r_2$ and $H + r_1$ and $H + r_2$ share the point $h_1 + r = h_2 + r_2$, then h_1 and h_2 differ by a rational number and therefore are equivalent. If $h_1 \neq h_2$, this contradicts the construction of H, which contains only one element from each equivalence class. If $h_1 = h_2$, then $r_1 = r_2$, which is again a contradiction.) Therefore, $(0, 1]$ is the union of the countably many disjoint sets $H + r$.

The sets $H + r$ for different r are "translations" of each other (they are all formed by starting from the set H and adding a number. The "uniform" probability measure (or Lebesgue measure) assigns a probability to each interval equal to its length, so that when an interval is a translation of another, they should have the same probability. We are interested in whether Lebesgue measure can be defined for all subsets of $(0, 1]$, while remaining translation-invariant. If this were possible, each set $H + r$ should have the same probability, and their probabilities should add to 1. But this is impossible, since there are infinitely many such sets.

A stronger statement is actually true, but harder to prove: there exists no probability measure on $((0, 1], 2^{(0,1]})$ under which $\mathbf{P}(\{x\}) = 0$ for all points x.

The Banach-Tarski Paradox. Let S be the two-dimensional surface of the unit sphere in three dimensions. There exists a subset F of S such that for any $k \geq 3$,

$$S = (\tau_1 F) \cup \cdots \cup (\tau_k F),$$

where each τ is a rigid rotation. For example, S can be made up by three rotated copies of F (suggesting probability equal to 1/3, but also by four rotated copies of F, suggesting probability equal to 1/4). Ordinary geometric intuition clearly fails when dealing with arbitrary sets.

References:

1. Billingsley, *Probability and Measure*, pp. 45-46.
2. Williams, *Probability with Martingales*, pp. 14-15, 192.