University of Illinois at Urbana-Champaign

ECE 534: RANDOM PROCESSES
Fall 2007
Midterm 2

Monday, November 12, 2007

Name: ____

- This is a closed-book exam. You may consult both sides of two sheets of notes, typed in font size 10 or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way email pagers, etc. may not be used.
- Write your answers in the space provided.
- Please show all of your work. Answers without appropriate justification will receive very little credit.

Score:
1. ____________ (12 points)
2. ____________ (12 points)
3. ____________ (12 points)

Total. ____________ (36 points)

Problem 1. Consider a random telegraph wave

$$X_t = X_0(-1)^{N_t}$$

where N_t is a Poisson process of rate λ and $X_0 \in \{-1, 1\}$ is a constant.

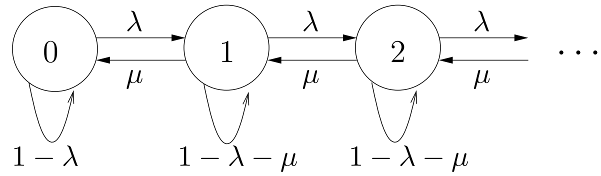
a) is X stationary?

b) is X a Markov process?

c) is X an independent increments process?

d) is X m.s. continuous?

Problem 2. Consider the following discrete-time birth-death Markov chain:



$$P(X_{t+1} = c | X_t = n) = \begin{cases} \lambda & c = n+1, \\ \mu, & c = n-1 \geq 0 \end{cases}$$

where $\lambda < \mu$ and $\lambda + \mu < 1$.

a) Use the Markov condition to show that for any Markov process,

$$P(X_{t-1} = x_{t-1} | X_t = x_t, X_{t+1} = x_{t+1} \ldots) = P(X_{t-1} = x_{t-1} | X_t = x_t).$$

b) Show for this specific chain that the equilibrium probability distribution satisfies

$$\pi_n \lambda = \pi_{n+1} \mu.$$

(*hint, use a proof by induction.*)

c) We say that a Markov chain is *reversible* if, in steady state, the backward running sequence of states is probabilistically indistinguishable from the forward running sequence, i.e. if

$$P\left(X_t = x_t | X_{t+1} = x_{t+1}, X_{t+2} = x_{t+2} \ldots\right) = P\left(X_{t+1} = x_t | X_t = x_{t+1}, X_{t-1} = x_{t+2} \ldots\right).$$

Show that for the birth-death chain defined above, X is indeed a reversible process.
(*hint: Use Bayes' rule. The solution does not require much space.*).

Problem 3. Consider a Poisson process N_t of rate λ and a Bernoulli process B (i.e. a sequence of i.i.d. binary random variables) of parameter $p = P(B_i = 1)$. Consider constructing two other point processes N_t^0 and N_t^1 by the following: at each epoch t_i of an arrival of N_t (i.e. where N_t jumps from $i-1$ to i), associate a jump at time t_i with N_t^0 if $B_i = 0$, otherwise associate a jump at time t_i with N_t^1 if $B_i = 1$.

a) Define $N^k(t, t+\delta) \triangleq N_{t+\delta}^k - N_t^k$. Find first-order approximations to $P(N^0(t, t+\delta) = 1)$ and $P(N^1(t, t+\delta) = 1)$ for small δ. Use your approximations to find

$$\lim_{\delta \to 0} \frac{P(N^0(t, t+\delta) = 1)}{\delta} \quad \text{and} \quad \lim_{\delta \to 0} \frac{P(N^1(t, t+\delta) = 1)}{\delta}.$$

b) Does N^0 have independent increments? Likewise, does N^1 have independent increments? Prove or provide a counterexample.

c) Characterize the full distribution of process N^0, and likewise for N^1. If N^0 and/or N^1 belong to a class of random processes discussed in the course, you need only describe the class of process and its associated parameters. Be sure to justify.

d) Are N^0 and N^1 independent random processes? Prove or give a counterexample.

Extra space if necessary. Please denote which problem you are using this extra space for.

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