University of Illinois at Urbana-Champaign

ECE 534: RANDOM PROCESSES
Fall 2007
Final Exam

Tuesday, December 11, 2007

Name: ____

- This is a closed-book exam. You may consult both sides of three sheets of notes, typed in font size 10 or equivalent handwriting size.
- Each part is worth 4 points.
- Calculators, laptop computers, Palm Pilots, two-way email pagers, etc. may not be used.
- Write your answers in the space provided.
- Please show all of your work. Answers without appropriate justification will receive very little credit.

Score:
1: _____________ (24 points)
2: _____________ (24 points)
3: _____________ (16 points)
4. _____________ (16 points)
Total: _____________ (80 points)

Problem 1. Note the following properties of the sinc function might be useful:

- $\text{sinc}(u) \triangleq \begin{cases} \frac{\sin(\pi u)}{\pi u}, & u \neq 0 \\ 1, & u = 0 \end{cases}$
- $\sum_j \text{sinc}\left(\frac{t_1 - jT}{T}\right) \text{sinc}\left(\frac{t_2 - jT}{T}\right) = \text{sinc}\left(\frac{t_2 - t_1}{T}\right)$
- For any integer $k \neq 0$: $\text{sinc}(kT) = 0$.

Consider a baseband random process $Z(t)$, bandlimited to $[-W, W]$, given by

$$Z(t) = \sum_j Z_j \text{sinc}\left(\frac{t - jT}{T}\right),$$

where $T = \frac{1}{2W}$ and the random variables $\{Z_j\}$ are i.i.d and $\mathcal{N}(0, 1)$.

a) Is Z a Gaussian random process?

b) Is Z a wide sense stationary (WSS) random process?

c) Is Z a stationary random process?

Now consider the random process W defined by

$$W(t) = \sum_j W_{2j}\text{sinc}\left(\frac{t-2jT}{T}\right) + W_{2j+1}\text{sinc}\left(\frac{t-(2j+1)T}{T}\right),$$

where the random variables indexed at even integer times, $\{W_{2j}\}$, are i.i.d. Gaussian $\mathcal{N}(0,1)$ and the random variables indexed at odd integer times, $\{W_{2j+1}\}$, are given by

$$W_{2j+1} = A_{2j}W_{2j},$$

where $\{A_{2j}\}$ are i.i.d. equiprobable ± 1 and independent of the random variables $\{W_{2j}\}$.

d) What is the probability distribution of W_{2j+1} for any j?

e) Is W a WSS process?

f) Is W a Gaussian process?

Problem 2.

a) Consider a Poisson process N with jumps given by times $T_1, T_2, \ldots$. Define $T_0 = t_0 = 0$. Show that for any integer $k \geq 1$, and any $t > t_{k-1}$:

$$P\left(T_k > t | T_{k-1} = t_{k-1}, \ldots, T_1 = t_1\right) = e^{-\lambda(t - t_{k-1})}.$$

(you need not write much here).

b) Suppose we are interested in characterizing the probability density of the jump times on $[0, T]$. Define

$$f_{N(T), T_1, \ldots, T_n}(n, t_1, \ldots, t_n) \triangleq \lim_{\Delta \to 0} \frac{P\left(T_1 \in [t_1, t_1 + \Delta], \ldots, T_n \in [t_n, t_n + \Delta], N(T) = n\right)}{\Delta^n}.$$

Use part a), along with a baby Bernoulli approach splitting $[0, T]$ into small intervals of length Δ, to show that

$$f_{N(T), T_1, \ldots, T_n}(n, t_1, \ldots, t_n) = \left[\prod_{i=1}^{n} \lambda e^{-\lambda(t_i - t_{i-1})}\right] e^{-\lambda(T - t_n)} = \lambda^n e^{-\lambda T}.$$

c) Use the Dirac delta function - along with the fact that on $[0, T]$, any sample path counting process N_t with n jumps is completely defined by the jump times $t_1, t_2, \ldots, t_n$ - to show that (b) can be expressed as:

$$f_{N(T),T_1,\ldots,T_n}(n, t_1, \ldots, t_n) = \exp\left\{\int_0^T \log \lambda dN_t - \lambda dt\right\}.$$

Now suppose we have a point process that no longer has the independent increments property that is special to Poisson. Rather than there being independent coin flips in $[t, t+\Delta]$, now the probability of heads of any coin flip in the interval $[t, t+\Delta]$ is allowed to depend upon the previous coin flips in $[0, t]$. Define the history at time t to be what has occurred up to time t: $H_t = \{N_\tau\}_{\tau<t}$. In this more general context, let us define the *conditional intensity function* $\lambda(t|H_t)$ as:

$$\lambda(t|H_t) \triangleq \lim_{\Delta\to 0} \frac{P\left(N(t, t+\Delta) = 1|H_t\right)}{\Delta}.$$

Let also assume, as for the Poisson case, that the process is regular: $\lim_{\Delta\to 0} \frac{P(N(t,t+\Delta)>1|H_t)}{\Delta} = 0$.

d) With this more general point process definition, show that

$$P\left(T_k > t_k|T_{k-1} = t_{k-1}, \ldots T_1 = t_1\right) = P\left(T_k > t_k|H_{t_{k-1}}\right) = \exp\left\{-\int_{t_{k-1}}^{t_k} \lambda\left(\tau|H_\tau\right) d\tau\right\}$$

$\Bigg($*hint: Just as for a fixed* $\beta > 0$, $\lim_{\Delta\to 0}\left[1 - \Delta\beta\right]^{t/\Delta} = \exp(-\beta t)$, *this can be generalized for an integrable function* $g(t) \geq 0$ *as* $\lim_{\Delta\to 0}\prod_{j=1}^{t/\Delta}\left[1 - \Delta g(j\Delta)\right] = \exp\left(-\int_0^t g(\tau)d\tau\right)$ $\Bigg)$.

e) Show that the probability density of the jump times on $[0, T]$ is given by

$$\begin{aligned} f_{N(T),T_1,\ldots,T_n}(n, t_1, \ldots, t_n) &= \left[\prod_{j=1}^{n} \lambda\left(t_j|H_{t_j}\right)\right] \exp\left\{-\int_0^T \lambda\left(t|H_t\right) dt\right\} \\ &= \exp\left\{\int_0^T \log \lambda\left(t|H_t\right) dN_t - \lambda\left(t|H_t\right) dt\right\}. \end{aligned}$$

f) Assume that the process N satisfies $\lambda(t|H_t) > 0$ for any t and any H_t. Define $T_0 = 0$. Show the *Time-Rescaling Theorem for Point Processes*, which states that the time-rescaled random variables $\{Z_i\}$, given by

$$Z_k = \int_{T_{k-1}}^{T_k} \lambda(t|H_t) dt,$$

form the inter-arrival times of a unit-rate Poisson process. *(Hint: show that for any integrable function $h > 0$, the function $g(x) = \int_0^x h(s)ds$ is monotonically increasing and thus invertible. Using this fact, equivalence of events, and careful reasoning involving the result of part a requires minimal space to show the result.)*

Problem 3. Suppose we observe Y which is related to X through multiplicative noise:

$$Y_t = X_t(1 + N_t)$$

where X and N are real, independent, and wide sense stationary (WSS) with $E[N_t] = 0$.

a) Show that Y is wide sense stationary (WSS).

Now suppose that $S_X(\omega) = \frac{2}{1+\omega^2}$ and $S_N(\omega) = \frac{4}{4+\omega^2}$.

b) Find the (possibly noncausal) Wiener filter for X_t.

c) Find the causal Wiener filter for X_t, given observations $\{Y_\tau\}_{\tau \leq t}$.

d) Suppose that X_t and N_t are jointly Gaussian random processes. Is Y_t a Gaussian process? Is Y_t a Gauss-Markov process?

Problem 4. Consider two jointly Wide Sense Stationary (jWSS) random processes Y and Z with power spectral densities S_Y, S_Z and cross-power spectral density S_{YZ}.

a) Suppose the processes Y and Z are put through a time-invariant linear transformation given by time-invariant functions h, g, k, l to produce the random processes U and V:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} h & k \\ g & l \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix}.$$

That is:

$$\begin{aligned} U_t &= \int_{-\infty}^{\infty} h(t-s) Y_s ds + \int_{-\infty}^{\infty} k(t-v) Z_v dv \\ V_\tau &= \int_{-\infty}^{\infty} g(\tau-a) Y_a da + \int_{-\infty}^{\infty} l(\tau-b) Z_b db. \end{aligned}$$

Show that

$$\begin{aligned} R_{UV}(t,\tau) &= h * \tilde{g} * R_Y(t-\tau) + k * \tilde{g} * R_{ZY}(t-\tau) + h * \tilde{l} * R_{YZ}(t-\tau) + k * \tilde{l} * R_Z(t-\tau) \\ R_U(t,\tau) &= h * \tilde{h} * R_Y(t-\tau) + h * \tilde{k} * R_{YZ}(t-\tau) + k * \tilde{h} * R_{ZY}(t-\tau) + k * \tilde{k} * R_Z(t-\tau) \\ R_V(t,\tau) &= g * \tilde{g} * R_Y(t-\tau) + g * \tilde{l} * R_{YZ}(t-\tau) + l * \tilde{g} * R_{ZY}(t-\tau) + l * \tilde{l} * R_Z(t-\tau), \end{aligned}$$

which equivalently means in the power spectral domain that

$$\begin{aligned} S_{UV} &= HG^* S_Y + KG^* S_{ZY} + HL^* S_{YZ} + KL^* S_Z \\ S_U &= |H|^2 S_Y + HK^* S_{YZ} + KH^* S_{ZY} + |K|^2 S_Z \\ S_V &= |G|^2 S_Y + GL^* S_{YZ} + LG^* S_{ZY} + |L|^2 S_Z. \end{aligned}$$

(*Hint: remember that* $\tilde{h}(v) \triangleq h^*(-v)$. *Also, you need not show the structure of both* R_U *and* R_V. *You can argue the other via symmetry.*).

(*more room for calculations for 4a... Remember - in case you are in a time crunch - every question part is only worth* 4 *points...*)

b) Suppose we would like to estimate a random process X that is jointly wide sense stationary (jWSS) with Y and Z. Consider using a linear estimator consisting of the LTI systems c and d of the form

$$\hat{X}_t = \int_{-\infty}^{\infty} c(t-s)Y_s + \int_{-\infty}^{\infty} d(t-v)Z_v dv$$

to optimally estimate X_t (possibly non-causally) from $\{Y_s\}_{\infty<s<\infty}$ and $\{Z_s\}_{\infty<s<\infty}$ in the linear MMSE sense. Find the functions c and d. Feel free to express this in terms of their Fourier transforms, $C(\omega)$ and $D(\omega)$.

c) Now suppose we would like to estimate X_t causally, using $\{Y_s\}_{s<t}$ and $\{Z_s\}_{s<t}$. Again, consider using LTI systems c and d (this time causal) to do so:

$$\hat{X}_t = \int_{-\infty}^{t} c(t-s)Y_s + \int_{-\infty}^{t} d(t-v)Z_v dv.$$

Use the orthogonality principle to express integral equations that must be satisfied on the LTI systems c and d. Also express this in the Fourier domain.

d) Now suppose that $S_Y = S_Z$ is rational and that $S_{YZ} = S_{ZY}$ is rational. Find the optimal causal filters $C(\omega)$ and $D(\omega)$. (*hint: consider taking an intermediate step, consider innovations sequences, and consider using the result of part a*).