ECE 534 RANDOM PROCESSES **SPRING 2005**
PROBLEM SET 7 **Due Wednesday, December 7**

Wiener Filtering of Random Processes

Assigned Reading: Chapter 7 of the notes.

1. A simple, noncausal estimation problem
Let $X = (X_t : t \in \mathbb{R})$ be a real valued, stationary Gaussian process with mean zero and autocorrelation function $R_X(t) = A^2 \text{sinc}(f_o t)$, where A and f_o are positive constants. Let $N = (N_t : t \in \mathbb{R})$ be a real valued Gaussian white noise process with $R_N(\tau) = \sigma^2 \delta(\tau)$, which is independent of X. Define the random process $Y = (Y_t : t \in \mathbb{R})$ by $Y_t = X_t + N_t$. Let $\widehat{X}_t = \int_{-\infty}^{\infty} h(t-s) Y_s ds$, where the impulse response function h, which can be noncausal, is chosen to minimize $E[D_t^2]$ for each t, where $D_t = X_t - \widehat{X_t}$.
(a) Find h.
(b) Identify the probability distribution of D_t, for t fixed.
(c) Identify the conditional distribution of D_t given Y_t, for t fixed.
(d) Identify the autocorrelation function, R_D, of the error process D, and the cross correlation function, R_{DY}.

2. On the MSE for causal estimation
Recall that if X and Y are jointly WSS and have power spectral densities, and if S_Y is rational with a spectral factorization, then the mean square error for linear estimation of X_{t+T} using $(Y_s : s \le t)$ is given by

$$\text{(MSE)} \;= R_X(0) - \int_{-\infty}^{\infty} \left| \left[\frac{e^{j\omega T} S_{XY}}{S_Y^-} \right]_+ \right|^2 \frac{d\omega}{2\pi}.$$

Evaluate and interpret the limits of this expression as $T \to -\infty$ and as $T \to \infty$.

3. Predicting the future of a simple WSS process
Let X be a mean zero, WSS random process with power spectral density $S_X(\omega) = \frac{1}{\omega^4 + 13\omega^2 + 36}$.
(a) Find the positive type, minimum phase rational function S_X^+ such that $S_X(\omega) = |S_X^+(\omega)|^2$.
(b) Let T be a fixed known constant with $T \ge 0$. Find $\widehat{X}_{t+T|t}$, the MMSE linear estimator of X_{t+T} given $(X_s : s \le t)$. Be as explicit as possible. (Hint: Check that your answer is correct in case $T = 0$ and in case $T \to \infty$).
(c) Find the MSE for the optimal estimator of part (b).

4. A singular estimation problem
Let $X_t = Ae^{j2\pi f_o t}$, where $f_o > 0$ and A is a mean zero complex valued random variable with $E[A^2] = 0$ and $E[|A|^2] = \sigma_A^2$. Let N be a white noise process with $R_N(\tau) = \sigma_N^2 \delta(\tau)$. Let $Y_t = X_t + N_t$. Let $\widehat{X}$ denote the output process when Y is filtered using the impulse response function $h(\tau) = \alpha e^{-(\alpha - j2\pi f_o)t} I_{\{t \ge 0\}}$.
(a) Verify that X is a WSS periodic process, and find its power spectral density (the power spectral density only exists as a generalized function–i.e. there is a delta function in it).
(b) Give a simple expression for the output of the linear system when the input is X.
(c) Find the mean square error, $E[|X_t - \hat{X}_t|^2]$. How should the parameter α be chosen to approximately minimize the MSE?

5. Estimation of a random signal, using the Karhunen-Loève expansion
Suppose that X is a m.s. continuous, mean zero process over an interval $[a, b]$, and suppose N is a white noise process, with $R_{XN} \equiv 0$ and $R_N(s,t) = \sigma^2\delta(s-t)$. Let $(\phi_k : k \geq 1)$ be a complete orthonormal basis for $L^2[a,b]$ consisting of eigenfunctions of R_X, and let $(\lambda_k : k \geq 1)$ denote the corresponding eigenvalues. Suppose that $Y = (Y_t : a \leq t \leq b)$ is observed.
(a) Fix an index i. Express the MMSE estimator of (X, ϕ_i) given Y in terms of the coordinates, $(Y, \phi_1), (Y, \phi_2), \ldots$ of Y, and find the corresponding mean square error.
(b) Now suppose f is a function in $L^2[a,b]$. Express the MMSE estimator of (X, f) given Y in terms of the coordinates $((f, \phi_j) : j \geq 1)$ of f, the coordinates of Y, the λ's, and σ. Also, find the mean square error.

6. Noiseless prediction of a baseband random process
Fix positive constants T and ω_o, suppose $X = (X_t : t \in \mathbb{R})$ is a baseband random process with one-sided frequency limit ω_o, and let $H^{(n)}(\omega) = \sum_{k=0}^{n} \frac{(j\omega T)^k}{k!}$, which is a partial sum of the power series of $e^{j\omega T}$. Let $\widehat{X}^{(n)}_{t+T|t}$ denote the output at time t when X is passed through the linear time invariant system with transfer function $H^{(n)}$. As the notation suggests, $\widehat{X}^{(n)}_{t+T|t}$ is an estimator (not necessarily optimal) of X_{t+T} given $(X_s : s \leq t)$.
(a) Describe $\widehat{X}^{(n)}_{t+T|t}$ in terms of X in the time domain. Verify that the linear system is causal.
(b) Show that $\lim_{n\to\infty} a_n = 0$, where $a_n = \max_{|\omega|\leq\omega_o} |e^{j\omega T} - H^{(n)}(\omega)|$. (This means that the power series converges uniformly for $\omega \in [-\omega_o, \omega_o]$.)
(c) Show that the mean square error can be made arbitrarily small by taking n sufficiently large. In other words, show that $\lim_{n\to\infty} E[|X_{t+T} - \widehat{X}^{(n)}_{t+T|t}|^2] = 0$.
(d) Thus, the future of a narrowband random process X can be predicted perfectly from its past. What is wrong with the following argument for general WSS processes? If X is an arbitrary WSS random process, we could first use a bank of (infinitely many) narrowband filters to split X into an equivalent set of narrowband random processes (call them "subprocesses") which sum to X. By the above, we can perfectly predict the future of each of the subprocesses from its past. So adding together the predictions, would yield a perfect prediction of X from its past.

7. Estimation given a strongly correlated process
Suppose g and k are minimum phase causal functions in discrete-time, with $g(0) = k(0) = 1$, and z-transforms $\mathcal{G}$ and $\mathcal{K}$. Let $W = (W_k : k \in \mathbb{Z})$ be a mean zero WSS process with $S_W(\omega) \equiv 1$, let $X_n = \sum_{i=-\infty}^{\infty} g(n-i)W_i$ and $Y_n = \sum_{i=-\infty}^{\infty} k(n-i)W_i$.
(a) Express R_X, R_Y, R_{XY}, $\mathcal{S}_X$, $\mathcal{S}_Y$, and $\mathcal{S}_{XY}$ in terms of g, k, $\mathcal{G}$, $\mathcal{K}$.
(b) Find h so that $\widehat{X}_{n|n} = \sum_{i=-\infty}^{\infty} Y_i h(n-i)$ is the MMSE linear estimator of X_n given $(Y_i : i \leq n)$.
(c) Find the resulting mean square error. Give an intuitive reason for your answer.