**ECE 534 RANDOM PROCESSES** **SPRING 2005**
**PROBLEM SET 5** **Due Wednesday, November 2**

**Basic Calculus of Random Processes**

**Assigned Reading:** Chapter 5 and Sections 8.3-8.5 of the notes.

**Problems to be handed in:**

**1. Properties of a binary valued process**
Let $Y = (Y_t : t \geq 0)$ be given by $Y_t = (-1)^{N_t}$, where $N$ is a Poisson process with rate $\lambda > 0$.
(a) Is $Y$ a Markov process? If so, find the transition probability function $p_{i,j}(s,t)$ and the transition rate matrix $Q$.
(b) Is $Y$ mean square continuous?
(c) Is $Y$ mean square differentiable?
(d) Does $\lim_{T\to\infty} \frac{1}{T}\int_0^T y_t dt$ exist in the m.s. sense? If so, identify the limit.

**2. Differentiation of the square of a Gaussian random process**
(a) Show that if random variables $(A_n : n \geq 0)$ are mean zero and jointly Gaussian and if $\lim_{n\to\infty} A_n = A$ $m.s.$, then $\lim_{n\to\infty} A_n^2 = A^2$ $m.s.$ (Hint: If $A, B, C$, and $D$ are mean zero and jointly Gaussian, then $E[ABCD] = E[AB]E[CD] + E[AC]E[BD] + E[AD]E[BC]$.)
(b) Show that if random variables $(A_n, B_n : n \geq 0)$ are jointly Gaussian and $\lim_{n\to\infty} A_n = A$ $m.s.$ and $\lim_{n\to\infty} B_n = B$ $m.s.$ then $\lim_{n\to\infty} A_n B_n = AB$ $m.s.$ (Hint: Use part (a) and the identity $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$.)
(c) Let $X$ be a mean zero, m.s. differentiable Gaussian random process, and let $Y_t = X_t^2$ for all $t$. Is $Y$ m.s. differentiable? If so, justify your answer and express the derivative in terms of $X_t$ and $X_t'$.

**3. A integral of white noise times an exponential**
Let $X = \int_0^t Z_u e^{-u} du$, where $Z$ is white Gaussian noise with autocorrelation function $\delta(\tau)\sigma^2$, for some $\sigma^2 > 0$.
(a) Find the autocorrelation function, $R_X(s,t)$ for $s, t \geq 0$.
(b) Is $X$ mean square differentiable? Justify your answer.
(c) Does $X_t$ converge in the mean square sense as $t \to \infty$? Justify your answer.

**4. A singular integral with a Brownian motion**
Consider the integral $\int_0^1 \frac{w_t}{t} dt$, where $w = (w_t : t \geq 0)$ is a standard Brownian motion. Since $\mathrm{Var}(\frac{w_t}{t}) = \frac{1}{t}$ diverges as $t \to 0$, we define the integral as $\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{w_t}{t} dt$ $m.s.$ if the limit exists.
(a) Does the limit exist? If so, what is the probability distribution of the limit?
(b) Similarly, we define $\int_1^\infty \frac{w_t}{t} dt$ to be $\lim_{T\to\infty}\int_1^T \frac{w_t}{t} dt$ $m.s.$ if the limit exists. Does the limit exist? If so, what is the probability distribution of the limit?

**5. A random process which changes at a random time**

Let $Y = (Y_t : t \in \mathbb{R})$ and $Z = (Z_t : t \in \mathbb{R})$ be stationary Gaussian Markov processes with mean zero and autocorrelation functions $R_Y(\tau) = R_Z(\tau) = e^{-|\tau|}$. Let $U$ be a real-valued random variable and suppose $Y$, $Z$, and $U$, are mutually independent. Finally, let $X = (X_t : t \in \mathbb{R})$ be defined by

$$X_t = \begin{cases} Y_t & t < U \\ Z_t & t \geq U \end{cases}$$

(a) Sketch a typical sample path of $X$.
(b) Find the first order distributions of $X$.
(c) Express the mean and autocorrelation function of $X$ in terms of the CDF, $F_U$, of $U$.
(d) Under what condition on $F_U$ is $X$ m.s. continuous?
(e) Under what condition on $F_U$ is $X$ a Gaussian random process?

**6. Karhunen-Loève expansion of a simple random process**

Let $X$ be a WSS random process with mean zero and autocorrelation function
$R_X(\tau) = 100(\cos(10\pi\tau))^2 = 50 + 50\cos(20\pi\tau)$.
(a) Is $X$ mean square differentiable? (Justify your answer.)
(b) Is $X$ mean ergodic in the m.s. sense? (Justify your answer.)
(c) Describe a set of eigenfunctions and corresponding eigenvalues for the Karhunen-Loève expansion of $(X_t : 0 \leq t \leq 1)$.

**7. Mean ergodicity of a periodic WSS random process**

Let $X$ be a mean zero periodic WSS random process with period $T > 0$. Recall that $X$ has a power spectral representation

$$X_t = \sum_{n \in \mathbb{Z}} \widehat{X}_n e^{2\pi jnt/T}.$$

where the coefficients $\widehat{X}_n$ are orthogonal random variables. The power spectral mass function of $X$ is the discrete mass function $p_X$ supported on frequencies of the form $\frac{2\pi n}{T}$, such that $E[|\widehat{X}_n|^2] = p_X(\frac{2\pi n}{T})$. Under what conditions on $p_X$ is the process $X$ mean ergodic in the m.s. sense? Justify your answer.